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A CHAIN REACTION PROCESS IN NUMBER THEORY

TRUMAN BOTTS, University of Virginia and COSRIMS

1. Introduction. If we set $n=1$ in the identity

$$\frac{1}{n}=\frac{1}{n+1}+\frac{1}{n(n+1)} \tag{1.1}$$

we have $1=1/2+1/2$. This expresses 1 as a sum of unit fractions, i.e., fractions with numerator 1; but they are not distinct unit fractions. Using the same identity with $n=2$ to replace one of these, we get $1=1/2+1/3+1/6$, which now expresses 1 as a sum of *distinct* unit fractions, the smallest of their denominators being 2. We may then use this last expression and a succession of substitutions from (1.1) to express 1 as a sum of distinct unit fractions in which the smallest denominator is 3, as follows:

$$\begin{aligned} 1 &= \frac{1}{2}+\frac{1}{3}+\frac{1}{6} \\ &= \left(\frac{1}{3}+\frac{1}{6}\right)+\frac{1}{3}+\frac{1}{6} \\ &= \left(\frac{1}{3}+\frac{1}{6}\right)+\left(\frac{1}{4}+\frac{1}{12}\right)+\left(\frac{1}{7}+\frac{1}{42}\right) \\ &= \frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{7}+\frac{1}{12}+\frac{1}{42}. \end{aligned}$$

The reader (with enough paper) may similarly obtain expressions for 1 as a sum of distinct unit fractions in which the smallest denominator is, successively, 4, 5, 6, 7. The question arises: can this process be continued indefinitely, so as to yield for each positive integer n an expression for 1 as a finite sum of distinct unit fractions, the smallest denominator being n? Or does the process at some point "blow up," leading to an endless succession of substitutions from (1.1)?

We shall show, in Theorem 2 below, that this process can indeed be continued indefinitely. In fact, Theorem 2 does more: it describes in quite detailed fashion the structure of the sequence of denominators obtained at each stage. Once we have established that the integer 1 is expressible as a finite sum of distinct unit fractions with arbitrarily large denominators, we see at once that there is a similar expression for any positive integer p: we simply think of p as the sum of p 1's. Having found such an expression for an arbitrary positive integer p, we may divide through by an arbitrary positive integer q and in this way obtain a similar expression for any positive rational number whatever. We thus arrive at the following theorem.

THEOREM 1. *For any positive rational number r and any positive integer n, there is an expression for r as a finite sum of distinct unit fractions, all with denominators greater than n.*

This is, in strengthened form, a result first obtained by Fibonacci [1] in the year 1202 and essentially rediscovered by Sylvester [2] in 1880. More recently this and allied results have been the subject of a number of investigations, among which we may mention [3], [4], [5], [6], [7], [8], [9], [10]. In the present note our interest is not so much in results like Theorem 1 but rather in the process itself, which certainly leads to a large class of unsolved problems. In order to delineate clearly some of these problems, as well as the process itself, we proceed in more general fashion below.

2. A chain reaction process. Consider two functions f_1 and f_2 assigning to each positive integer n the respective values $f_1(n)$ and $f_2(n)$ which are integers greater than n. Suppose we are given a finite sequence R_1 of distinct positive integers, listed in order of increasing size:

$$r_1, r_2, \cdots, r_q.$$

From this list we delete r_1 and insert at appropriate points $f_1(r_1)$ and $f_2(r_1)$. The resulting list may or may not contain repetitions. If it contains no repetitions, the process stops and we define R_2 as the sequence listed. If it does contain one or more repetitions, we delete from the list one of the occurrences of the first such repeated numbers r and insert in the list the numbers $f_1(r)$ and $f_2(r)$. Again, if the resulting list contains no repetitions, the process stops and we define R_2 as the sequence of numbers listed. If, however, it does contain repetitions, we again delete from the list one of the occurrences of the first repeated number and insert in the list the values of the functions f_1 and f_2 at this number; and so forth. Either this "chain reaction" process ultimately terminates and thus defines a finite sequence R_2, or else it is "self-sustaining," i.e., it continues without end.

It is easy to give an example where this latter happens. Define f_1, f_2, and R_1 by setting, for each positive integer n,

$$\begin{aligned} f_1(n) &= n + 1 \\ f_2(n) &= n + 2 \\ R_1 &= (1, 2). \end{aligned} \tag{2.1}$$

Our process begins by deleting 1 from the list

$$1, 2$$

of terms of the sequence R_1, and inserting the numbers $f_1(1) = 2$ and $f_2(1) = 3$ in this list, so that the new list becomes

$$2, 2, 3.$$

Since this contains a repetition, we delete from the list one of the occurrences of the first (and, in this case, only) repeated number 2, and we insert in the list the numbers $f_1(2) = 3$ and $f_2(2) = 4$. The new list is then

$$2, 3, 3, 4,$$

which again contains a repetition. It is evident that this process continues without end, producing the succession of lists

$$2, 3, 4, 4, 5$$
$$2, 3, 4, 5, 5, 6$$
$$2, 3, 4, 5, 6, 6, 7$$

and so on.

In cases where our process does terminate, however, we arrive at a second stage where we have obtained a finite sequence R_2 of distinct terms. We may now repeat the above process, using R_2 in place of R_1: from the sequence R_2 we delete its first number r, then insert $f_1(r)$ and $f_2(r)$, look for repetitions, etc. In this way we may also succeed in reaching a third stage, at which we have generated from R_2 a new finite sequence R_3. Proceeding from stage to stage in this fashion, we may in fact find it possible to continue indefinitely generating finite sequences $R_1, R_2, R_3, \cdots$, of distinct positive integers.

This last is just what occurs in the case where f_1, f_2, and R_1 are defined by

$$\begin{aligned} f_1(n) &= n+1 \\ f_2(n) &= n(n+1) \qquad (2.2) \\ R_1 &= (1) \end{aligned}$$

as we show in Theorem 2, below. Note that with the choices of f_1, f_2, and R_1 of (2.2) our process becomes essentially the one discussed in Section 1, when we focus attention on the sequences of denominators concerned.

3. Preliminaries and lemmas. The application of our process to (2.2) begins by deleting 1 from the (one-long) list of terms of R_1 and inserting $f_1(1)=2$ and $f_2(1)=2$. We then delete one of the two resulting occurrences of 2 and insert $f_1(2)=3$ and $f_2(2)=6$. The resulting list

$$2, 3, 6$$

contains no repetitions, so we have reached the second stage and we define this sequence to be R_2.

Below we summarize the first few stages, underlining repeated numbers rather than writing them out twice.

R_1: 1
$\underline{2}$
R_2: 2, 3, 6
$\underline{3}, \underline{6}$
3, 4, $\underline{6}$, 12
R_3: 3, 4, 6, 7, 12, 42
$\underline{4}$, 6, 7, $\underline{12}$, 42
4, 5, 6, 7, $\underline{12}$, 20, 42

R_4: 4, 5, 6, 7, 12, 13, 20, 42, 156

$\underline{5}$, 6, 7, 12, 13, $\underline{20}$, 42, 156

5, $\underline{6}$, 7, 12, 13, $\underline{20}$, 30, 42, 156

5, 6, $\underline{7}$, 12, 13, $\underline{20}$, 30, $\underline{42}$, 156

5, 6, 7, 8, 12, 13, $\underline{20}$, 30, $\underline{42}$, 56, 156

5, 6, 7, 8, 12, 13, 20, 21, 30, $\underline{42}$, 56, 156, 420

R_5: 5, 6, 7, 8, 12, 13, 20, 21, 30, 42, 43, 56, 156, 420, 1806

At any stage—say the nth—at which a finite sequence R_n is obtained, we may think of this sequence as composed of blocks of consecutive numbers of the natural number sequence. It is not hard to see that there is always an initial such block beginning with n itself and that all other blocks at this stage necessarily begin with numbers of form $k(k+1)>n$.

For instance, in R_4 there is an initial block 4, 5, 6, 7 of length 4 beginning with 4 itself, a block 12, 13 of length 2 beginning with $12=3\cdot 4$, and three blocks of length 1 beginning with $20=4\cdot 5$, $42=6\cdot 7$, and $156=12\cdot 13$ respectively.

For use in Theorem 2 it will be convenient to define recursively a sequence $L(1), L(2), \cdots, L(k), \cdots$, by setting $L(1)=2$, and

$$L(k) = \sum_{2\leqq t(t+1)\leqq k} L(t)$$

for all $k\geqq 2$. We shall also make use of the estimate in the following lemma.

LEMMA 1. $L(2)=2$, *and for each integer* $k\geqq 3$ *we have* $L(k)<k$.

To prove this we first observe that it is certainly true for $k=2, 3, \cdots, 11$, since we may verify directly from the definition that $L(2)=\cdots=L(5)=2$ and $L(6)=\cdots=L(11)=4$. Proceeding inductively, consider any $k\geqq 11$, such that for all integers k_1 with $3\leqq k_1\leqq k$ we have $L(k_1)<k_1$. The proof will be complete if we can show that $L(k+1)<k+1$.

Now there is a positive integer q such that $q(q+1)\leqq k+1<(q+1)(q+2)$; and since $k\geqq 11$, we see that $q\geqq 3$. In fact, $3\leqq q<q(q+1)\leqq k+1$, so $3\leqq q\leqq k$, and hence by the induction hypothesis we have

$$L(q) < q. \tag{3.1}$$

Also $3\leqq (q-1)q<q(q+1)\leqq k+1$, so $3\leqq (q-1)q\leqq k$, and again, by the induction hypothesis we have

$$L((q-1)q) < (q-1)q. \tag{3.2}$$

Now since $q(q+1)\leqq k+1<(q+1)(q+2)$, we have $L(k+1)=L(q(q+1))$, and from the definition of L we see that $L(q(q+1))=L((q-1)q)+L(q)$. Using these facts and (3.1) and (3.2), we find that

$$\begin{aligned} L(k+1) &= L(q(q+1)) = L((q-1)q) + L(q) < (q-1)q + q \\ &= q^2 < q(q+1) \leqq k+1, \end{aligned}$$

completing the proof.

Theorem 2 will also involve the following recursively defined notion. For all positive integers k, the $(k, 1)$-*train* of blocks (of consecutive integers) is defined to consist of the single block of length 1 beginning with $k(k+1)$. Now consider any positive integer B. Proceeding recursively, suppose that for all positive integers $B' < B$ and all integers $k' \geqq B'$ we have already defined the (k', B')-train. Then for every integer $k \geqq B$ the (k, B)-*train* of blocks is defined to consist of

(1) a block of length B beginning with $k_1 = k(k+1)$,
(2) all the blocks of the $(k_1, B-1)$-train (if $B > 1$),
(3) all the blocks of the $(k_1+1, B-2)$-train (if $B > 2$),

⋮

(B) all the blocks of the $(k_1+B-2, 1)$-train.

Note that all the blocks of the (k, B)-train are necessarily separated from one another, at least if we grant that they all begin with distinct numbers. This is because they all begin with numbers of form $k'(k'+1)$ with $k' \geqq k$, so that the interval from one such beginning number to the next possible one is at least $(k+1)(k+2) - k(k+1) = 2(k+1)$, whereas all the blocks of the train have lengths $\leqq B \leqq k$.

To see that all blocks of the (k, B)-train *do* begin with distinct numbers, we first note that only the block described under (1) begins with $k_1 = k(k+1)$ while all the others begin with larger numbers of the form $(k_r+u)(k_r+u+1)$, where k_r is itself such a beginning number and $0 \leqq u < B$. Supposing two of these coincide, consider the smallest such, say

$$(k_r + u)(k_r + u + 1) = (k_s + v)(k_s + v + 1). \tag{3.3}$$

We see that then necessarily $k_r + u = k_s + v$, whence

$$|k_r - k_s| = |v - u| < B \leqq k.$$

But since k_r and k_s are each of the form $k'(k'+1)$ with $k' \geqq k$, and since they differ by less than k, they can only coincide, contradicting the choice of the numbers in (3.3) as the smallest such.

We shall need two more lemmas.

LEMMA 2. *Where k and B are positive integers such that $k < B$, suppose that the numbers in the (k, B)-train are listed in order and that the first number $k_1 = k(k+1)$ is listed twice. The application of our chain reaction process (with f_1 and f_2 as in (2.2)) to this list then yields a list whose blocks are precisely those of the $(k, B+1)$-train.*

For the proof, it is clear that the chain reaction process will propagate itself through the initial block of the (k, B)-train, thus lengthening it by 1 and so producing the initial block of the $(k, B+1)$-train. At the same time the numbers $k_1(k_1+1)$, $(k_1+1)(k_1+2)$, $\cdots$, $(k_1+B-1)(k_1+B)$ are produced. All but the last of these produce repetitions of the first numbers in, respectively, the

$(k_1, B-1)$-train, the $(k_1+1, B-2)$-train, $\cdots$, the $(k_1+B-2, 1)$-train, while the last constitutes the $(k_1+B-1, 1)$-train.

Now the lemma is evident for $B=1$. Proceeding inductively, we may suppose it holds for all (k', B')-trains for which $B'<B$ (and $k'>B'$). In particular, then, it holds for the $(k_1, B-1)$-train, the $(k_1+1, B-2)$-train, $\cdots$, the $(k_1+B-2, 1)$-train. This completes the proof.

LEMMA 3. *Suppose that a block of the (k, B)-train and a block of the (k', B')-train have a number in common, and say that $k \leqq k'$. Then $k'(k'+1)$ is the beginning number of some block of the (k, B)-train.*

For the proof, first note that for these blocks with a common number the beginning number of one must belong to the other. Now as noted earlier, because of the restriction on its length, each block of a train contains just *one* number of form $r(r+1)$, namely its beginning number. Therefore these two blocks with a common number must have a common *beginning* number. We maintain that the smallest such common beginning number (of a block of the (k, B)-train and a block of the (k', B')-train) must be $k'(k'+1)$, the smallest beginning number of a block of the (k', B')-train. For otherwise, by an argument similar to that in the paragraph containing (3.3), there would have to be an even smaller such common beginning number, a contradiction.

4. The principal theorem. We now have the following theorem.

THEOREM 2. *For each $n=1, 2, \cdots$, the above-described chain reaction process, when applied to (2.2), yields at the nth stage a finite sequence R_n. When $n>2$, the sequence R_n consists of blocks of consecutive numbers as follows:*

(i) *an initial block beginning with n, ending with some number e_n for which there is a $k<n$ with $(k-1)k<e_n \leqq k(k+1)-2$, and having length $L(e_n)$;*

(ii) *for each $k<n$ for which $k(k+1)>e_n+1$, the blocks of the $(k, L(k))$-train;*

(iii) *for each k in the initial block for which $B_{k,n} \equiv L(k)-(k+1-n)>0$, the blocks of the $(k, B_{k,n})$-train.*

Furthermore, for each k with $n \leqq k \leqq e_n$ we have $B_{k,n} \geqq 0$.

Before proceeding to the proof, let us note that in view of Lemma 1 we have $L(k) \leqq k$ and $B_{k,n}<k$, so that the trains in (ii) and (iii) are certainly well defined.

We also maintain that the blocks described under (i), (ii), and (iii) of Theorem 2 are all separated from one another. That is, no two of these blocks intersect or abut each other. Toward establishing this, we first note that the block described under (i) is separated from all the blocks described under (ii) and (iii), since e_n+1 is smaller than the beginning numbers of all these other blocks. By the argument following the definition of (k, B)-train, for each given train described under (ii) or (iii) the blocks of that train are all separated from one another.

Now consider any two trains under (ii), say the $(k, L(k))$-train and the $(k', L(k'))$-train, where $k<k'<n$. Then, where $k_1=k(k+1)$,

$$n < k_1 < k'(k'+1) < n(n+1) < k_1(k_1+1). \tag{4.1}$$

Now the first block of the $(k, L(k))$-train begins with $k_1 = k(k+1)$, while all its other blocks begin with numbers at least as large as $k_1(k_1+1)$. Therefore by (4.1) no block of the $(k, L(k))$-train can begin with $k'(k'+1)$; and hence by Lemma 3 the blocks of the $(k, L(k))$-train are all disjoint from those of the $(k', L(k'))$-train. Furthermore, no block of the $(k, L(k))$-train could abut a block of the $(k', L(k'))$-train. For, by the argument following the definition of (k, B)-train, each block of a train begins with a number of form $r(r+1)$ and has length $\leqq r$, whereas the next possible such beginning number is $(r+1)(r+2)$. Thus the blocks of the $(k, L(k))$-train are all separated from those of the $(k', L(k'))$-train.

Next, consider any train under (ii), say the $(k, L(k))$-train, where $k < n$, and consider any train under (iii), say the $(k', B_{k',n})$-train, where $n \leqq k' \leqq e_n < k(k+1) < n(n+1)$. Then, where $k_1 = k(k+1)$, we have

$$k_1 < n(n+1) \leqq k'(k'+1) < k_1(k_1+1). \tag{4.2}$$

Using (4.2) in place of (4.1), we now argue, exactly as we did for two trains under (ii), that the blocks of the $(k, L(k))$-train are all separated from those of the $(k', B_{k',n})$-train.

Finally, consider any two trains under (iii), say the $(k, B_{k,n})$-train and the $(k', B_{k',n})$-train, where $n \leqq k < k' \leqq e_n < n(n+1)$. Then, where $n_1 = n(n+1)$ and $k_1 = k(k+1)$,

$$n_1 \leqq k_1 < k'(k'+1) < n_1(n_1+1) \leqq k_1(k_1+1). \tag{4.3}$$

Using (4.3) in place of (4.1), we again argue as before that the blocks of the $(k, B_{k,n})$-train are all separated from those of the $(k', B_{k',n})$-train. This completes the proof that all the blocks described under (i), (ii), and (iii) of Theorem 2 are separated.

5. Proof of the theorem. We have already verified that for $n = 1, 2, 3, 4, 5$ the process yields finite sequences R_n. In particular, we have verified that for $n = 3$ the sequence R_n is

$$3, 4, 6, 7, 12, 42.$$

We now check that the blocks of consecutive numbers in this sequence are as described in (i), (ii), and (iii). First, there is an initial block

$$3, 4$$

beginning with $n = 3$ and ending with $e_n = 4$, a number for which the required inequality in (i) holds with $k = 2 < n$. Thus (i) holds. There is only one k, namely $k = 2$, which satisfies the restrictions in (ii). Corresponding to $k = 2$ we have the blocks of the $(2, L(2))$-train: a block of length $L(2) = 2$ beginning with $2 \cdot 3 = 6$ and a block of length $L(2) - 1 = 1$ beginning with $6 \cdot 7 = 42$. Thus (ii) holds. Finally, for $k = 3$, which is in the initial block, we have $B_{3,3} = L(3) - (3+1-3) = 1 > 0$. Corresponding to this we have the blocks of the $(3, B_{3,3})$-train, which reduces to a single block of length $B_{3,3} = 1$ beginning with $3 \cdot 4 = 12$. For $k = 4$, also in the initial block, we have $B_{4,3} = L(4) - (4+1-3) = 0$, corresponding to which there are no additional blocks. Thus (iii) holds. The final statement in Theorem 2 is easily verified for $n=3$.

The assertions of the theorem have now been completely verified for $n=1, 2, 3$. The reader can readily verify them for $n=4$ and $n=5$ as well. Proceeding inductively, let us suppose that the assertions of the theorem hold for all values up through some $n \geqq 3$. We wish to show that they then hold with n replaced by $n+1$.

The first step in trying to pass from stage n to stage $n+1$ consists of deleting the first term, n, from the sequence R_n and inserting at appropriate points in the sequence $n+1$ and $n(n+1)$. This produces a repetition of $n+1$, which is already in the initial block, so we next delete one occurrence of $n+1$ and insert $n+2$ and $(n+1)(n+2)$. It is clear that our chain reaction process does propagate itself in this way through the entire initial block. Thus the initial block of R_n is in effect shifted one to the right. Now there is at least one k satisfying the restrictions of (ii), and in fact the smallest such k is that of (i). For this smallest k there is by (ii), a block of length $L(k)$ beginning with $k(k+1)$, namely the first block of the $(k, L(k))$-train. We see that this block is the second block of R_n. Now, by (i), either $e_n < k(k+1)-2$ or else $e_n = k(k+1)-2$.

In case $e_n < k(k+1)-2$ we have $e_n+1 < k(k+1)-1$. This says that the initial block of R_n, when shifted one to the right, still stops short of the second block of R_n. Therefore in this case $e_{n+1}=e_n+1$, and so $(k-1)k < e_{n+1} \leqq k(k+1)-2$ for the k of (i). From this and the definition of L we see that $L(e_{n+1})=L(e_n)$. Thus in this case (i) holds with n replaced by $n+1$ (the new k being the same as the old k).

In case $e_n = k(k+1)-2$ we have $e_n+1 = k(k+1)-1$, so that the initial block of R_n, when shifted one to the right, now coalesces with the second block of R_n. As a result, e_{n+1} is now the ending number of this second block, and the length of the initial block of R_{n+1} is the sum of the lengths of the initial block and the second block of R_n, namely

$$
\begin{aligned}
L(e_n) + L(k) &= \sum_{2 \leq t(t+1) \leq e_n} L(t) + L(k) \\
&= \sum_{2 \leq t(t+1) \leq e_{n+1}} L(t) = L(e_{n+1}).
\end{aligned}
\tag{5.1}
$$

Now by Lemma 1 we have $0 < e_{n+1} - k(k+1) = L(k) - 1 < k$, so that

$$
\begin{aligned}
k(k+1) < e_{n+1} &< k(k+1) + k \\
&= k(k+2) \leqq (k+1)(k+2) - 2.
\end{aligned}
\tag{5.2}
$$

Thus in this case too, (i) holds with n replaced by $n+1$ (the new k being the old $k+1$). We have now completely verified (*i*) with n replaced by $n+1$.

By tracing our process step by step through the entire initial block we have dealt with all possible repeated numbers that can have been produced up to $n(n+1)$, a number produced at the very first step. Taking $k=n$ in (iii) we see that $B_{n,n}=L(n)-1>0$ so that in R_n we already have the blocks of the $(n, L(n)-1)$-train. Thus the first number of this train, $n(n+1)$, is now a repeated number. Hence by Lemma 2 our chain reaction process converts this

$(n, L(n)-1)$-train of R_n into the $(n, L(n))$-train of R_{n+1}. This establishes (ii) with n replaced by $n+1$, at least for the value $k=n<n+1$. And as far as smaller values of k are concerned, a moment's reflection shows that the assumption of (ii) for stage n yields (ii) for stage $n+1$. Thus (ii) is established with n replaced by $n+1$.

It will be convenient, before establishing (iii) with n replaced by $n+1$, to complete the inductive proof of the final statement in Theorem 2. That is, supposing that n is such that for each k with $n \leqq k \leqq e_n$ we have $B_{k,n} \geqq 0$, we wish to show that for each k with $n+1 \leqq k \leqq e_{n+1}$ we have

$$B_{k,n+1} \geqq 0. \tag{5.3}$$

We start from the fact that by (i) there is an $r<n$ such that $(r-1)r<e_n \leqq r(r+1)-2$.

Case I. $(r-1)r<e_n<r(r+1)-2$. Then $e_{n+1}=e_n+1$, and for each k with $n+1 \leqq k \leqq e_n=e_{n+1}-1$ we have $B_{k,n} \geqq 0$ by the induction hypothesis, whence $B_{k,n+1}=B_{k,n}+1 \geqq 1$. Also

$$\begin{aligned} B_{e_{n+1},n+1} &= B_{e_n+1,n+1} \\ &= L(e_n + 1) - (e_n + 1 + 1 - (n + 1)) \\ &= L(e_n) - (e_n + 1 - n) = 0. \end{aligned} \tag{5.4}$$

This completes the proof of (5.3) in this case.

Case II. $e_n=r(r+1)-2$. Then $e_n+1=r(r+1)-1$, and so

$$e_{n+1} = e_n + 1 + L(r) = r(r + 1) - 1 + L(r). \tag{5.5}$$

First, for any k with $n+1 \leqq k<e_n+1=r(r+1)-1$, we have $B_{k,n+1}=B_{k,n}+1 \geqq 1$ just as in Case I. When $k=e_n+1$, we have $B_{e_n+1,n+1}=0$ as in (5.4). Finally, if $r(r+1) \leqq k \leqq e_{n+1}$, we see from the definition of L and the fact that $e_n+1 =r(r+1)-1$ that $L(k)=L(e_n+1)+L(r)$. Using this fact and (5.5) we find that

$$\begin{aligned} B_{k,n+1} &= L(k) - (k + 1 - (n + 1)) \\ &\geqq L(e_n + 1) + L(r) - (e_n + 1 + L(r) + 1 - (n + 1)) \\ &= L(e_n + 1) - (e_n + 1 + 1 - (n + 1)) \\ &= 0. \end{aligned} \tag{5.6}$$

as in (5.4), completing the proof of (5.3).

It remains only to establish (iii) with n replaced by $n+1$. Consider, for each $k>n$ in the initial block of R_n, the number $k(k+1)$ produced by the propagation of our process through the initial block. If $B_{k,n}>0$, this $k(k+1)$ is a repetition of the initial number of the $(k, B_{k,n})$-train of R_n described in (iii). By Lemma 2 and the fact that $B_{k,n}+1=B_{k,n+1}$, our process converts this $(k, B_{k,n})$-train into the $(k, B_{k,n+1})$-train. The final statement in Theorem 2 shows that the only other possibility is $B_{k,n}=0$. In this case $B_{k,n+1}=B_{k,n}+1=1$, and the single number $k(k+1)$ itself constitutes the $(k, B_{k,n+1})$-train. Furthermore, the computation in (5.4) shows that $B_{e_n+1,n+1}=0$. Thus for all $k=n+1$,

$\cdots, e_n+1$ we have established (iii) with n replaced by $n+1$. In Case I, above, where $e_{n+1}=e_n+1$, this completes the proof.

In Case II, where $e_n=r(r+1)-2$ and (5.5) holds, take any k such that $r(r+1)\leqq k\leqq e_{n+1}$. Recall that by (5.2) (with "k" replaced by "r") we have

$$r(r+1) < e_{n+1} \leqq (r+1)(r+2) - 2.$$

Hence we see that $L(k)=L(e_{n+1})$, which is the length, $e_{n+1}-n$, of the new initial block. Thus by (5.5)

$$L(k) = e_{n+1} - n = r(r+1) - 1 + L(r) - n,$$

and hence we easily see that

$$B_{k,n+1} = L(k) - (k + 1 - (n+1)) = L(r) - (k - r(r+1)) - 1.$$

Thus we have

$$\begin{aligned} B_{r(r+1),n+1} &= L(r) - 1, \\ B_{r(r+1)+1,n+1} &= L(r) - 2, \\ &\vdots \\ B_{r(r+1)+L(r)-1,n+1} &= B_{e_{n+1},n+1} = 0. \end{aligned} \tag{5.7}$$

Now by (ii) we already have, at stage n, the blocks of the $(r, L(r))$-train. Apart from the block of length $L(r)$ beginning with $r(r+1)$ (which is now a part of the initial block at stage $n+1$), these consist of:

all the blocks of the $(r(r+1), L(r)-1)$-train,

all the blocks of the $(r(r+1)+1, L(r)-2)$-train,

$\vdots$

all the blocks of the $(r(r+1)+L(r)-2, 1)$-train.

This, together with the equations (5.7), completes the establishment of (iii) with n replaced by $n+1$, for those k having $r(r+1)\leqq k\leqq e_{n+1}$. The proof is complete.

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PACKING CYLINDERS INTO CYLINDRICAL CONTAINERS

SIDNEY KRAVITZ, Dover, New Jersey

The problem of determining the minimum radius of the cylindrical container which can contain N equal cylinders is important in packaging and in the design of rope and conductor cables. A few general results and a few special cases have been proved [1, 2]. The rest is empirical.

Figures 2 to 17 inclusive show good ways to pack cylinders into cylindrical containers for $N \leqq 19$. Readers are invited to find smaller containers than those given here.

To assist those who would like to try their hand at this problem, a do-it-yourself kit of formulas and tables is presented. First it is noted that if (ρ_1, θ_1) and (ρ_2, θ_2) are the polar coordinates of the centers of two cylinders of unit radius then they can be packed without interference if

$$\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos(\theta_1 - \theta_2) \geqq 4.$$

The equality holds when the cylinders touch. Second, if we pack N cylinders into an annular ring as shown in Figure 1, then

$$\frac{R}{r} = \frac{2}{1 - \tan^2\left(\frac{\pi}{4} - \frac{\pi}{2N}\right)}$$

where R is the outer radius of the ring and r is the radius of each equal cylinder inside the annulus. (R/r) is given in Table I, column 2, as a function of N. If we place an additional cylinder, A, external to the ring, but touching two of the cylinders inside the annulus, then the furthermost point, R_0, of the external cylinder is given by

$$\frac{R_0}{r} = \frac{1}{\tan \frac{\pi}{N}} + (\sqrt{3} + 1).$$

For an internal cylinder, B, the innermost point, R_1, is given by

$$\frac{R_1}{r} = \frac{1}{\tan \frac{\pi}{N}} - (\sqrt{3} + 1).$$

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$$\frac{R_1}{r} = \frac{1}{\tan \frac{\pi}{N}} - (\sqrt{3} + 1).$$

(R_0/r) and (R_1/r) are given in Table I, columns 3 and 4, as functions of N.

Imagine hexagons which circumscribe the cylinders packed within the container. The packing efficiency is defined as the area of these hexagons divided by the area of the container:

$$\phi = \frac{2\sqrt{3}}{\pi} \cdot \frac{N}{(R/r)^2} = 1.10266 \frac{N}{(R/r)^2},$$

where R is the radius of the container. The efficiency cannot exceed one. Efficiencies are presented in column 6, Table I, and in Figures 2 to 17 inclusive. $N=7$ and $N=19$ give high efficiencies.

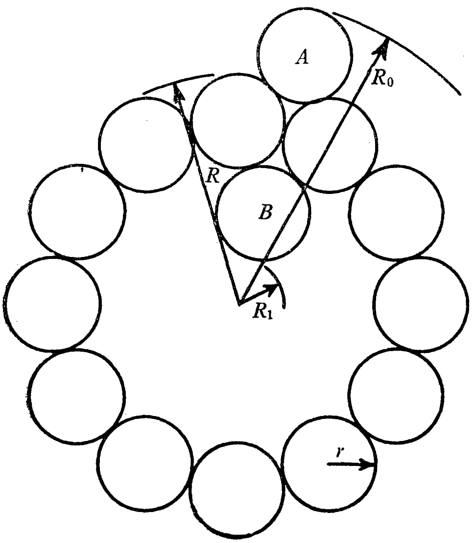

FIG. 1.

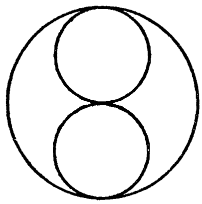

FIG. 2. $N=2$, $R/r=2$, $\phi=.5513$.

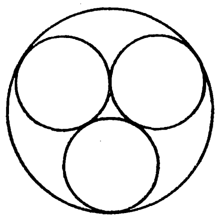

FIG. 3. $N=3$, $R/r=1+\frac{2}{\sqrt{3}}=2.1547$, $\phi=.7125$.

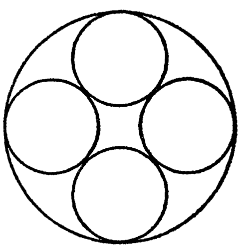

FIG. 4. $N=4$, $R/r=1+\sqrt{2}=2.4142$, $\phi=.7568$.

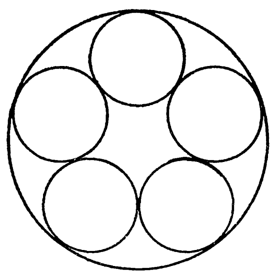

FIG. 5. $N=5$, $R/r=1+\sqrt{\frac{(10+2\sqrt{5})}{5}}=2.7013$, $\phi=.7555$.

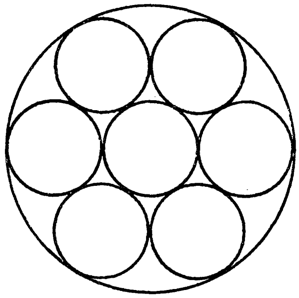

FIG. 6. $R/r=3$ for $N=6$, $\phi=.7351$
for $N=7$, $\phi=.8576$.

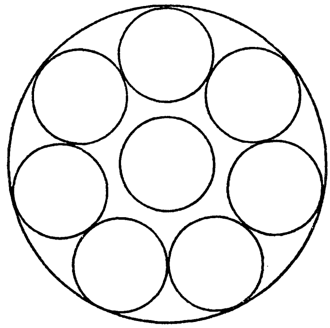

FIG. 7. $N=8$, $R/r=1+\csc(\pi/7)=3.3046$
$\phi=.8078$.

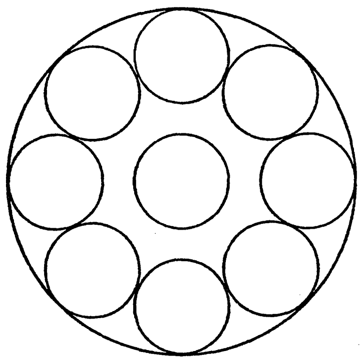

FIG. 8. $N=9$, $R/r=1+\sqrt{(4+2\sqrt{2})}=3.6131, \phi=.7602$.

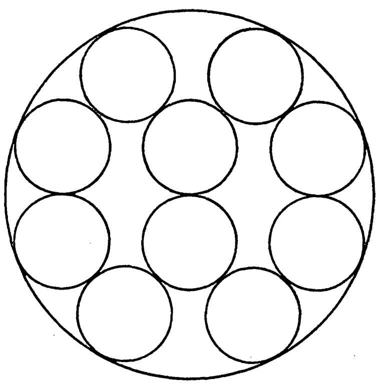

FIG. 9. $N=10$, $R/r=1+2\sqrt{2}=3.8284$, $\phi=.7523$.

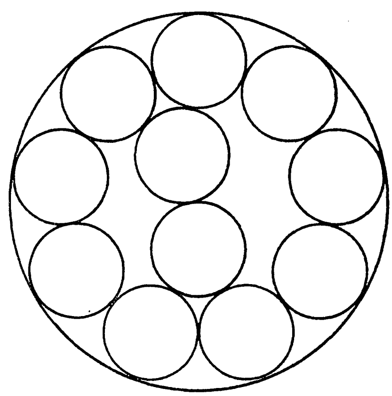

FIG. 10. $N=11$, $R/r=1+\csc 20°=3.9238$, $\phi=.7879$.

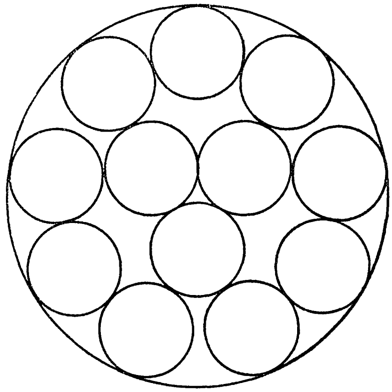

FIG. 11. $N=12$, $R/r=4.0294$, $\phi=.8150$.

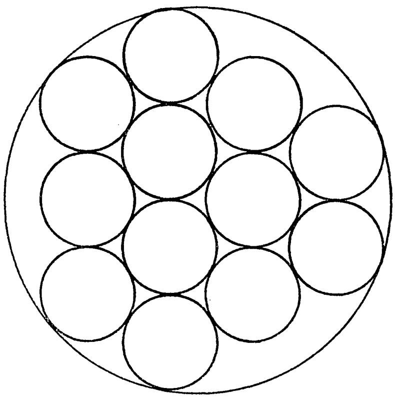

FIG. 12. $N=12$, $R/r=4.0550$. This is *not* the minimum solution for $N=12$. See Figure 11.

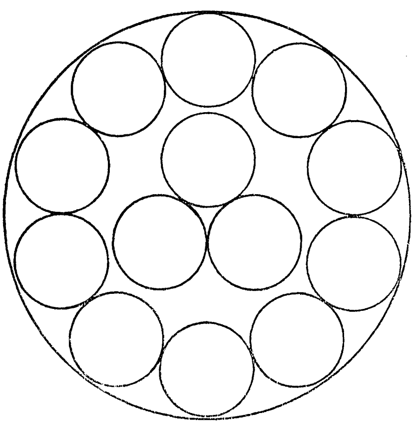

FIG. 13. $N=13$, $R/r=2+\sqrt{5}=4.2361$, $\phi=.7989$.

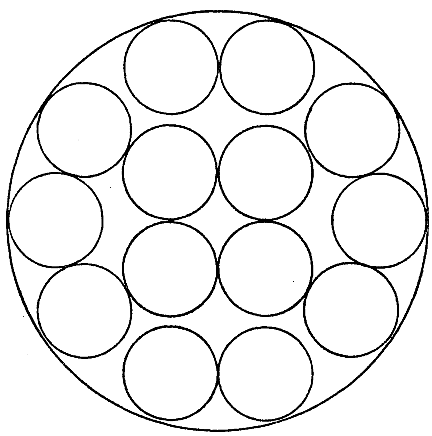

FIG. 14. $N=14$, $R/r=4.3738$, $\phi=.8069$.

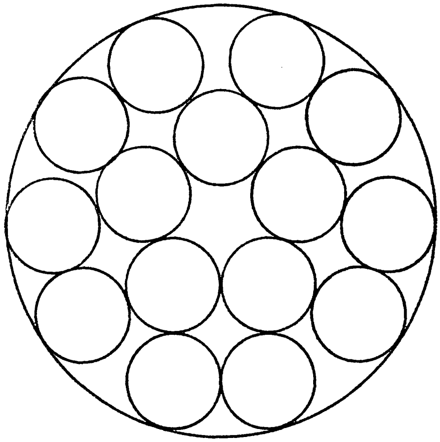

FIG. 15. $N = 15$, $R/r = 1 + \sqrt{1 + \left\{2 + \left(\sqrt{\frac{5+1}{4}}\right)\left(\sqrt{\frac{10+2\sqrt{5}}{5}}\right)\right\}^2}$

$= 4.5213$, $\phi = .8091$.

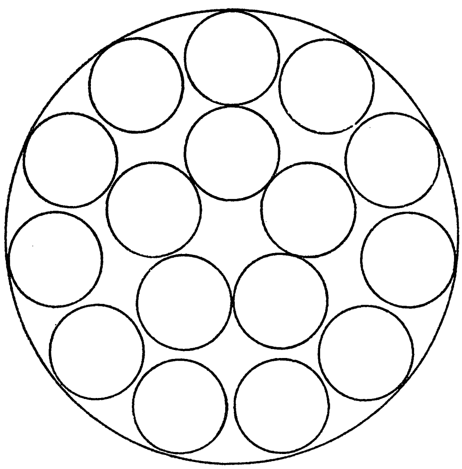

FIG. 16. $N=16$, $R/r=4.7013$, $\phi=.7982$. This is not the minimum solution. It is possible to squeeze the outer ring in a bit.

TABLE I.

N	Annular Ring (Figure 1)			Designs Presented in Figures 2 to 17 Inclusive	
	R/r	R_0/r	R_1/r	R/r	Efficiency, ϕ
2				2.0000	.5513
3	2.15470	3.30940		2.1547	.7125
4	2.41421	3.73205		2.4142	.7568
5	2.70130	4.10843		2.7013	.7555
6	3.00000	4.46410		3.0000	.7351
7	3.30477	4.80857		3.0000	.8576
8	3.61313	5.14626		3.3046	.8078
9	3.92381	5.47953	0.01543	3.6131	.7602
10	4.23607	5.80973	0.34563	3.8284	.7523
11	4.54947	6.13774	0.67364	3.9238	.7879
12	4.86370	6.46410	1.00000	4.0294	.8150
13	5.17858	6.78921	1.32511	4.2361	.7989
14	5.49396	7.11334	1.64923	4.3738	.8069
15	5.80974	7.43668	1.97258	4.5213	.8091
16	6.12583	7.75939	2.29529	4.7013	.7982
17	6.44219	8.08158	2.61748	4.8637	.7924
18	6.75877	8.40333	2.93923	4.8637	.8390
19	7.07554	8.72472	3.26062	4.8637	.8857
20	7.39246	9.04580	3.58170	Column 5	Column 6
21	7.70951	9.36662	3.90252		
22	8.02668	9.68720	4.22310		
23	8.34395	10.00759	4.54349		
24	8.66130	10.32780	4.86370		
25	8.97874	10.64787	5.18376		
26	9.29624	10.96779	5.50369		
27	9.61380	11.28760	5.82350		
28	9.93141	11.60730	6.14319		
29	10.24908	11.92690	6.46280		
30	10.56678	12.24641	6.78231		
31	10.88453	12.56585	7.10175		
32	11.20231	12.88522	7.42112		
33	11.52012	13.20452	7.74042		
34	11.83797	13.52377	8.05967		
35	12.15583	13.84296	8.37886		
36	12.47373	14.16210	8.69800		
37	12.79164	14.48120	9.01710		
38	13.10958	14.80025	9.33615		
39	13.42754	15.11927	9.65517		
40	13.74551	15.43825	9.97415		
41	14.06350	15.75720	10.29310		
42	14.38151	16.07612	10.61202		
43	14.69953	16.39501	10.93091		
44	15.01756	16.71388	11.24977		
45	15.33561	17.03272	11.56861		
46	15.65367	17.35153	11.88743		
47	15.97174	17.67033	12.20623		
48	16.28982	17.98910	12.52500		
49	16.60790	18.30786	12.84375		
50	16.92600	18.62659	13.16249		
Column 1	Column 2	Column 3	Column 4		

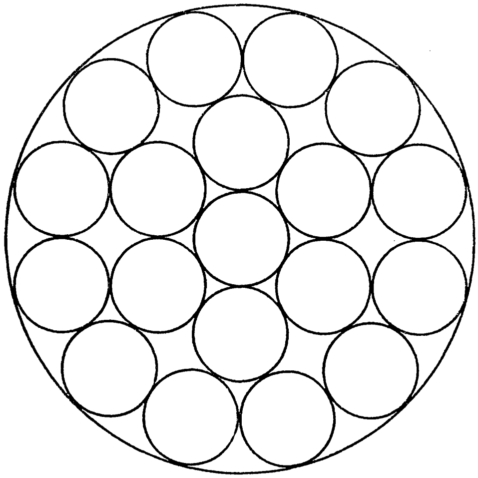

FIG. 17. $R/r=1+\sqrt{2}+\sqrt{6}=4.8637$
for $N=17$, $\phi=.7924$
$N=18$, $\phi=.8390$
$N=19$, $\phi=.8857$.

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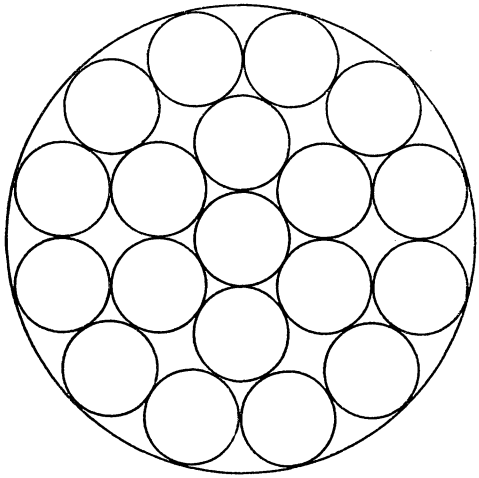
TOPOLOGY AND ANALYSIS

R. C. BUCK, University of Wisconsin

1. Introduction. In what follows, I speak as an analyst, not a topologist. In particular, I am not discussing topology as an axiomatic structure, nor yet as a touchstone to bring young minds to life, nor as an organized body of theorems. Rather, I shall confine my remarks largely to the role of topology as something which illuminates topics in analysis, and which provides a more geometric viewpoint. I will give a series of unconnected illustrations.

2. Elementary calculus. To "illuminate" can mean to clarify or simplify. Suppose we assume a background of the simplest anatomy of topology—acquaintance with the meaning of open, closed, connected, compact, etc., all applied chiefly in the context of sets in Euclidean space.



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2. Elementary calculus. To "illuminate" can mean to clarify or simplify. Suppose we assume a background of the simplest anatomy of topology—acquaintance with the meaning of open, closed, connected, compact, etc., all applied chiefly in the context of sets in Euclidean space.

Here are three statements about sets and continuous functions:

(1) The set $\{\text{all } (x, y) \text{ with } x^2 - xy + y^3 \leqq 1\}$ is a closed set.

(2) If f is a continuous real valued function defined on $[a, b]$, then f is bounded there and achieves a maximum and minimum at points of $[a, b]$.

(3) If f is a continuous real valued function defined on $[a, b]$, and $f(a) < 0$ and $f(b) > 0$, then for some choice of t, $a < t < b$, one has $f(t) = 0$.

Let us also consider three general statements of a topological nature:

(1)′ If E is a closed set, and F is a continuous mapping, then the inverse image $F^{-1}(E)$ is closed.

(2)′ Continuous mappings send compact sets into compact sets.

(3)′ Continuous mappings send connected sets into connected sets.

The connection between these triples should be very clear; each statement $(A)'$ implies the statement (A), and indeed, throws light upon it, supplying a clearer insight into the basic truth of these statements.

3. Intermediate calculus. Sometimes analysis can assist in putting over topological ideas. When one is first introducing topological concepts, a frequent tactic is to present a student with several different topologies on a space, and invite him to compare their properties. If the underlying set is the plane or a subset of the plane, then many of these alternate topologies may strike a student as contrived. He may be left with the feeling that the discussion of these topologies and their properties is of little permanent significance. This is especially true if he has already learned that there is a very definite sense in which the Euclidean topology is the *only* (normed) topology on the plane (or n-space) (see [2], pp. 191–192).

Perhaps this discussion of the comparative properties of several topologies on the same set could better be done in an intermediate analysis course. Suppose we select $\mathfrak{F}$ as the class of all real valued functions defined on $X = (-\infty, \infty)$. Then, it is readily seen that many of the standard topics in analysis really deal with the comparison between two competing limit topologies defined on $\mathfrak{F}$:

(i) pointwise convergence: $\{f_n\} \to f$ iff $f_n(x) \to f(x)$ for each $x \in X$.
(ii) uniform convergence: $\{f_n\} \to f$ iff $f_n(x) \to f(x)$, uniformly for all $x \in X$.

Each of these is important in its own right, and each plays a role in analysis. Thus it is not an artificial question to ask for differences in the properties of these topologies.

As an illustration, let $\mathfrak{C}$ be the subset of $\mathfrak{F}$ consisting of all the continuous functions. Then we can ask, in each case: Is $\mathfrak{C}$ a closed set? Note that the answers differ, and that, furthermore, the results convey in different language certain central results about convergence of sequences of functions.

Other questions might be asked. What does a compact set look like? Does

$$F(\phi) = \int_0^1 \phi, \quad \phi \in \mathfrak{C}$$

define a continuous function on $\mathfrak{C}$? on $\mathfrak{F}$? Is the set $\mathcal{P}$ of polynomials a dense subset of $\mathfrak{C}$?

In each case, a natural topological question yields an important analytic result.

4. Advanced analysis. Another very important aspect of topology can also be discussed at this stage. (We also correct a possibly misleading inference in the preceding section.)

The notion of pointwise convergence (i) gives rise to the "pointwise" topology defined on $\mathfrak{F}$. A neighborhood $\mathfrak{N}$ of a function $f_0 \in \mathfrak{F}$ is specified by a finite number of points $x_1, \cdots, x_N$ and associated positive numbers $\delta_1, \delta_2, \cdots, \delta_N$, and $\mathfrak{N} = \mathfrak{N}(x_1, \cdots, x_N; \delta_1, \cdots, \delta_N)$ consists of all functions $f \in \mathfrak{F}$ such that $|f(x_j) - f_0(x_j)| < \delta_j$ for $j = 1, 2, \cdots, N$.

This can then be shown to be a topology which is *not* metrizable, even though it arises in a very natural manner; in it, the closure of a set *cannot* be obtained merely by adjoining all limits of converging sequences. [Thus, the notion of pointwise convergence, (i) does not define a true topology on $\mathfrak{F}$, although uniform convergence (ii) does!]

To show that this is the case, let $\mathcal{S}$ be the subset of $\mathfrak{F}$ consisting of all continuous functions f which satisfy the pair of conditions

$$\int_0^1 f(x)dx = 1 \quad \text{and} \quad 0 \leqq f(x) \leqq 2, \quad \text{for all } x, \ -\infty < x < \infty. \tag{4}$$

It is easy to see that the constant function 0 belongs to the pointwise closure of $\mathcal{S}$—i.e., every neighborhood $\mathfrak{N}(x_1, x_2, \cdots, x_N; \delta_1, \cdots, \delta_N)$ about 0 contains functions in $\mathcal{S}$. However, there cannot exist any sequence $f_n \in \mathcal{S}$ which converges pointwise to 0. For, if $f_n(x) \to 0$ for each $x \in [0, 1]$, then by the Lebesgue Bounded Convergence Theorem [1], we should have to have

$$\int_0^1 f_n \to \int_0^1 0 = 0 \tag{5}$$

which contradicts condition (4). (There is, of course, a *net* $\{f_\alpha\}$ in $\mathcal{S}$ which converges pointwise to 0.) (See [2], pp. 143–146.)

In a different direction, there are examples in which a topological viewpoint brings additional enlightenment. A standard theorem in elementary analysis is the continuity of the inverse of a monotonic continuous function on a bounded interval. Here is a simple alternate approach. We first investigate the relationship between the continuity of a function and the nature of its graph.

THEOREM. *Let A and B be metric spaces, with A compact, and let $f: A \to B$ be a function. Let $G = \{$all $(a, f(a))$ for $a \in A\}$ be its graph in $A \times B$. Then f is continuous iff G is a compact set.*

Proof. One direction is trivial. If f is continuous, so is the mapping $x \xrightarrow{T} (x, f(x))$, and $G = T(A)$ must be compact. Conversely, if G were compact but f not continuous, then there would exist a sequence $\{x_n\}$ in A convergent to $x_0 \in A$, but with $d(f(x_n), f(x_0)) \geqq \delta > 0$ for all n. The sequence $p_n = (x_n, f(x_n))$ lies in G, and must therefore have a subsequence $\{p_{n_j}\}$ converging to a point $(a, b) \in G$.

Clearly, $a=x_0$ so that $b=f(x_0)$, and therefore $f(x_{n_j})\to f(x_0)$, which contradicts the assumed property of $\{x_n\}$.

[*Query.* Can you remove the hypothesis that A and B are *metric* spaces?]

An immediate trivial corollary of this is then the following basic result.

COROLLARY. *Let A and B be metric spaces, with A compact. Let $f: A\to B$ be continuous and one-to-one. Then f^{-1} is continuous.*

Proof. Let $B_0=f(A)$, a compact set in B. Then, $f^{-1}: B_0\to A$ is a function whose graph is homeomorphic to the graph of f. Since f is continuous, this set is compact, and f^{-1} is continuous.

At this point, one can point out the importance of the requirement that A be compact. Let $\mathcal{C}$ be the metrizable space consisting of all continuous real valued functions on $[0, 1]$ with the uniform convergence topology. Define a function $F: \mathcal{C}\to\mathcal{C}$ by $F(\phi)=\psi$ where $\psi(x)=\int_0^x\phi$, $0\leqq x\leqq 1$. It is easily checked that F is continuous. (Indefinite integration of a uniformly convergent sequence leaves it uniformly convergent, if we normalize at one point.) It is also one-to-one, for if $F(\phi_1)=F(\phi_2)$ then $\int_0^x(\phi_1-\phi_2)=0$ for all $x\in[0, 1]$, and $\phi_1=\phi_2$. However, F^{-1}—which exists as a function—is *not* continuous! (One sees that F^{-1} is D, the differentiation operator, and if $\psi_n\to\psi_0$ uniformly, it does not follow that ψ_n' converges to ψ_0'.)

Here, then, is a convenient example of a function F which is continuous, and whose graph is therefore closed, but whose graph is not compact. Likewise, D is a function whose graph is closed, but which is not continuous on $\mathcal{C}$. (A simpler example is the function g on $[0, 1]$ defined by $g(x)=1/x$, $g(0)=1$.).

An invited address at the MAA meeting in Denver, 1965.

References

1. W. Rudin, Principles of Mathematical Analysis, McGraw-Hill, New York, 1964, pp. 246–247.
2. A. Wilansky, Functional Analysis, Blaisdell, New York, 1964.

SIDE-AND-DIAGONAL NUMBERS

FREDERICK V. WAUGH and MARGARET W. MAXFIELD,
Arlington, Virginia and Gainesville, Florida

1. Purpose of this paper. Our purpose is not to present new methods. It is to discuss methods that were developed over 2,000 years ago. We hope that our presentation may be interesting, and that it may help some readers to understand the nature of irrational surds. But we do not claim any new discoveries. If there is anything new in this paper, it is in the presentation.

2. Side-and-diagonal numbers. The ancient Greeks had an ingenious way of approximating square roots. The Pythagoreans proved that the square root of 2 is "irrational,"—meaning that it is not the ratio between any two integers.

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Euclid meant the same thing when he said that the diagonal of a square is "incommensurable with its side."

Since $\sqrt{2}$ is irrational, its exact value cannot be stated by any fraction. The best we can do in any number system is an approximation. Many of the early Greeks (including Aristarchus of Samos) used 7/5 as an approximation to $\sqrt{2}$. If the sides of a square were 5 inches, the diagonal would be $\sqrt{50}$ inches, or $\sqrt{(7^2+1)}$ inches,—or slightly more than 7 inches. That is what Plato [7, p. 309] meant when he said that 7 is "the rational diameter of 5", and that $\sqrt{50}$ is "the irrational diameter of 5".

But 7/5 is only a rough approximation to $\sqrt{2}$. Greek arithmeticians sought closer estimates. Since

$$d/s \neq \sqrt{2}, \text{ i.e., } d^2/s^2 \neq 2, \text{ i.e., } d^2 \neq 2s^2, \tag{1}$$

they sought solutions of the equation

$$d^2 - 2s^2 = \pm 1, \text{ i.e., } d^2 = 2s^2 \pm 1, \text{ i.e., } d^2/s^2 = 2 \pm 1/s^2. \tag{2}$$

This equation is a special case of "Pell's equation" $d^2 - ns^2 = N$. (See, for instance, [6] pp. 158–161.)

Clearly, the larger the numbers s and d satisfying (2), the closer d^2/s^2 is to 2, and the closer d/s is to $\sqrt{2}$. The Pythagoreans developed a systematic way of getting all the positive solutions of (2). This will be explained in the next section.

Probably the Greeks used similar methods to find close approximations to the square roots of many other numbers. When Archimedes [3, pp. 91–98] set out to estimate the value of π, he needed lower and upper bounds to $\sqrt{3}$. He simply stated, as Heath [3, p. lxxx] says, "without a word of explanation," that

$$265/153 < \sqrt{3} < 1351/780. \tag{3}$$

In the same work, Archimedes gave bounds to the square roots of seven large numbers—again without explanation. Perhaps Archimedes assumed that the Greek methods of bounding square roots was well known. And perhaps the Greeks used several methods. But, as we shall indicate in the next section of this paper, we know that one method was to form a sequence of fractions

$$d_0/s_0,\ d_1/s_1,\ d_2/s_2,\ \cdots,\ d_t/s_t, \tag{4}$$

such that

a. each fraction could be computed from the one immediately preceding it, and

b. each successive fraction was a closer approximation to $\sqrt{n}$, —preferably alternating from greater than to less than $\sqrt{n}$.

The Greek mathematicians and philosophers were especially interested in $\sqrt{2}$, the ratio of the diagonal of a square to its side. Thus, in the sequence of fractions for $\sqrt{2}$, the numerator was known as the diagonal number and the denominator as the side number. In later years this general method of approximating the square root of any number became known as the "method of side-and-diagonal numbers," even though the geometric relations were different.

Günther [2] and Heath [3] have surveyed the literature concerning side-and-diagonal numbers. Günther's review is an exhaustive survey of all the literature on the subject up to 1882. Heath's treatment is shorter, but penetrating. Also, it is part of a most interesting discussion of Greek methods of arithmetic, in general. We shall here review the work of three men who have contributed greatly to the subject. Then we shall present a synthesis.

3. Theon of Smyrna. About 130 A.D., Theon of Smyrna wrote a book explaining Plato's mathematics. Theon of Smyrna is distinguished from Theon of Alexandria, who also was a mathematician, and who wrote about square roots. Dupuis [1, pp. 71–75], in 1892, translated Theon's book into French. That book is our best source of information about how the ancient Greeks used side-and-diagonal numbers to approximate the square root of 2.

Theon gave the following sequence of fractions

$$1/1,\ 3/2,\ 7/5,\ 17/12,\ \cdots \tag{5}$$

each term of which is an approximation to $\sqrt{2}$, and each term of which can be computed from the previous term by the recursion formula

$$d_{t+1}/s_{t+1} = \frac{2s_t + d_t}{s_t + d_t}\,. \tag{6}$$

One can easily continue the sequence. The next fraction would be $(2\cdot 12+17)/(12+17)=41/29$.

Theon discussed the sequence of fractions (5) and stated that $1^2=2\cdot 1^2-1$, $3^2=2\cdot 2^2+1$, $7^2=2\cdot 5^2-1$, and $17^2=2\cdot 12^2+1$. He concluded that, in general,

$$d_t^2 = 2\cdot s_t^2 \pm 1;\ 2 = (d_t^2 \mp 1)/s_t^2;\ \sqrt{2} = \sqrt{(d_t^2 \mp 1)}/s_t, \tag{7}$$

for any positive integer t. He also noted that the sign before the 1 alternated from $-$ to $+$, and he stated that the ratio d/s approached $\sqrt{2}$ as t increased.

Why the initial fraction 1/1? Theon said it is because unity is the principle of all figures and the generator of all numbers. That is Platonic metaphysics. Actually, one could start with any numbers d_0 and s_0 with $d_0^2-2s_0^2=e_0$. Then, using (7), he would get $d_t^2-2s_t^2=(-1)^te_0$. If d_0 and s_0 are integers, the smallest possible e_0 is ± 1. The modern mathematician would be satisfied with any initial fraction d/s, such that $d^2-2s^2=\pm 1$. He would accept 1/1, because $1^2-2\cdot 1^2=-1$.

Theon did not prove equation (7); he only showed that it held for the first several terms. Equation (7) can be proved by induction. From (6), $d_{t+1}^2=4s_t^2+4s_td_t+d_t^2$ and $2s_{t+1}^2=2s_t^2+4s_td_t+2d_t^2$ so that

$$d_{t+1}^2 - 2s_{t+1}^2 = 2s_t^2 - d_t^2 = -(d_t^2 - 2s_t^2). \tag{8}$$

Since, as Theon showed, the first several fractions in the sequence are such that $d_t^2-2s_t^2=\pm 1$, with the sign before the 1 alternating from $+$ to $-$, this will continue indefinitely, and since $d_t^2-2s_t^2=\pm 1$,

$$d_t^2/s_t^2 = 2 \pm 1/s_t^2, \tag{9}$$

for each positive integer t. As t increases, s_t increases, and its reciprocal approaches zero. Thus, d_t/s_t continuously approaches $\sqrt{2}$, and is alternately below and above the true value, as Theon said.

To anticipate further results, Theon's system can be written in terms of vectors and matrices. Let

$$v_0 = [s_0 \quad d_0] = [1 \quad 1] \quad \text{and} \quad T = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}. \tag{10}$$

Then

$$e_0 = 1^2 - 2 \cdot 1 = |T| = -1. \tag{11}$$

The sequence of fractions is

$$v_1 = v_0 T,\ v_2 = v_1 T = v_0 T^2, \cdots, v_{t+1} = v_t T = v_0 T^t, \quad \text{and} \quad e_t = (-1)^{t+1}. \tag{12}$$

To get faster convergence, one could use

$$T^2 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}.$$

Here $|T^2| = 9 - 8 = +1$. If he started with $v_0 = [1 \quad 1]$, he would get

$$v_1 = [5 \quad 7],\ v_2 = [29 \quad 41],\ v_3 = [169 \quad 239], \cdots.$$

In other words, the sequence would be 1/1, 7/5, 41/29, 239/169, $\cdots$, which approaches $\sqrt{2}$ monotonically from below. Similarly, if he started with $v_0 = [1\ 2]$, he would get a sequence approaching $\sqrt{2}$ monotonically from above.

A more rapidly converging sequence that oscillates about $\sqrt{2}$ can be had from

$$T^3 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^3 = \begin{bmatrix} 7 & 10 \\ 5 & 7 \end{bmatrix},$$

with $|T^3| = |T|^3 = (-1)^3 = -1$. Again, starting with $v_0 = [1 \quad 1]$, one would find

$$v_1 = [12 \quad 17],\ v_2 = [169 \quad 239], \cdots$$

from which one would know that $17/12 > \sqrt{2} > 239/169$.

Before leaving Theon of Smyrna, we note in passing that the sequence 1/1, 3/2, 7/5, 17/12, $\cdots$ can also be obtained from the continued fraction

$$\sqrt{2} = 1 + \frac{1}{2+}\frac{1}{2+}\frac{1}{2}\cdots, \tag{13}$$

since

$$1 = 1/1,\ 1 + \tfrac{1}{2} = \tfrac{3}{2},\ 1 + \frac{1}{2+}\frac{1}{2} = 7/5, \cdots.$$

We think that the ancient Greek method is far simpler than continued fractions.

4. de Lagny. M. de Lagny's paper [5] was written in 1723 and published in 1725. (Referees must have been slow even then!) It is still one of the most penetrating analyses of the uses of side-and-diagonal numbers.

Early in his paper (p. 57) de Lagny remarked that in approximating the square root of 2 "the best that any finite intelligence can do is to find *regularly, indefinitely, and without any fumbling* (notice these three conditions, and especially the last) is, I say, to find the sequence of all squares taken two by two, such that the difference between the larger and double the smaller be as small as possible, whether in excess or in default, and it is evident that this difference can not be less than unity."

Thus de Lagny says that the problem of approximating $\sqrt{2}$ is to find the sequence d_t/s_t, such that $d_t^2=2s_t^2\pm 1$, —and to find it by a regular process that can be continued indefinitely, and that does not require any *tâtonnement* (which we take to mean no fumbling around, no trial-and-error, no guessing).

He suggests that Archimedes might have found his bounds $265/153<\sqrt{3}<1351/780$ by using two sequences of fractions. The first sequence is

$$2/1,\ 7/4,\ 26/15,\ 97/56,\ 362/209,\ 1351/780,\ \cdots, \tag{14}$$

each term of which is greater than $\sqrt{3}$, and closer to $\sqrt{3}$ than the term immediately preceding it. The second sequence is

$$1/1,\ 5/3,\ 19/11,\ 71/41,\ 265/153,\ 980/571,\ \cdots, \tag{15}$$

each term of which is less than $\sqrt{3}$, and closer to $\sqrt{3}$ than the term immediately preceding it.

In either sequence (14) or (15) the terms after the first are

$$\frac{d_{t+1}}{s_{t+1}}=\frac{3s_t+2d_t}{2s_t+d_t}, \tag{16}$$

or

$$v_{t+1}=\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}v_t. \tag{17}$$

The sequence of fractions in (14) extended indefinitely by the recursion formula (16) gives all the positive solutions of $d^2-3s^2=+1$. The sequence in (15) gives all the positive solutions of $d^2-3s^2=-2$. To prove this, note that $d_{t+1}^2=9s_t^2+12s_td_t+4d_t^2$ and $3s_{t+1}^2=12s_t^2+12s_td_t+3d_t^2$ so that $d_{t+1}^2-3s_{t+1}^2=-3s_t^2+d_t^2$. This result is due to the fact that

$$\begin{vmatrix}2 & 3\\ 1 & 2\end{vmatrix}=+1.$$

Thus, if $e_t=d_t^2-3s_t^2$, $e_{t+1}=e_t=e$, and e stays constant as d_t and s_t increase. We see easily from the first term of (14) that $e=+1$. The first term in (15) shows that for each fraction in that sequence $e=-2$. It can be proved by the Greek method that there are not two integers s and d such that $d^2-3s^2=-1$.

The above development also proves convergence. Since the error term, e, is fixed for either sequence, we have $d_t^2-3s_t^2=e$, and $d_t^2/s_t^2=3+e/s_t^2$. But s_t increases with t, so d_t^2/s_t^2 approaches 3, and d_t/s_t approaches $\sqrt{3}$.

Several commentators have noted that it seems strange that Archimedes would have carried the first of de Lagny's sequences to 6 terms to get the very close bound $\sqrt{3}<1351/780$, but would have carried the second sequence to only 5 terms to get the much looser bound $\sqrt{3}>265/153$. They have searched for a single sequence in which 265/153 and 1351/780 would be successive terms. More on this later.

After discussing his two sequences of fractions approximating $\sqrt{3}$, de Lagny proceeded to discuss the general case of $\sqrt{n}$. He gave initial fractions and recursion formulas for approximating the square roots of 2, 3, 5, 6, 7, 8, 13, 41, and 43. He then described a general process for approximating $\sqrt{n}$. Let $a^2<n<b^2$, where a^2 and b^2 are fairly close approximations to n. (By trial squaring one can usually approximate to the nearest integer easily.) Let

$$s_0 = 1,\ d_0 = b,\ s_{t+1} = as_t + d_t,\ d_{t+1} = ns_t + ad_t. \tag{18}$$

This can be written

$$v_0 = [1 \quad b],\ T = \begin{bmatrix} a & n \\ 1 & a \end{bmatrix},\ |T| = a^2 - n < 0. \tag{19}$$

De Lagny noted that one can use either a or b for d_0 and either a or b for the diagonal entries of T, and so obtain different sequences. For a sequence with $|T| = a^2 - n < 0$, as in (19), d_t/s_t oscillates around $\sqrt{n}$ as it approaches $\sqrt{n}$. For a sequence with $|T| = b^2 - n > 0$, d_t/s_t increases monotonically toward $\sqrt{n}$ if $d_0/s_0 = a/1 < \sqrt{n}$, or decreases monotonically toward $\sqrt{n}$ if $d_0/s_0 = b/1 > \sqrt{n}$.

Note that for $t>0$, $d_{t+1}^2 = n^2s_t^2 + 2ans_td_t + a^2d_t^2$ and $ns_{t+1}^2 = na^2s_t^2 + 2ans_td_t + nd_t^2$ so that

$$d_{t+1}^2 - ns_{t+1}^2 = (a^2 - n)(d_t^2 - ns_t^2) = (a^2 - n)^t(b^2 - n). \tag{20}$$

Thus

$$d_t^2/s_t^2 = n + (a^2 - n)^{t-1}(b^2 - n)/s_t^2. \tag{21}$$

If the fraction at the right in (21) approaches zero as t increases, d_t/s_t approaches $\sqrt{n}$.

We note that de Lagny's sequence is useful for approximating the square roots of large numbers. Early in his "Measurement of a Circle" Archimedes [**3**, p. 94] stated that $\sqrt{(349450)} > 591\ 1/8$. Try de Lagny's principle. The surd in question is less than 600 and greater than 590. Let $s_0=1$, $d_0=b=600$, or $v_0 = [1\ 600]$, and

$$T = \begin{bmatrix} a & n \\ 1 & a \end{bmatrix} = \begin{bmatrix} 590 & 349450 \\ 1 & 590 \end{bmatrix}.$$

As the determinant of this T is negative, we see from (21) that d_t^2/s_t^2 oscillates around n. Therefore, since $\sqrt{(349450)} < d_0/s_0 = 600/1$, we have $\sqrt{(349450)} > d_1/s_1 = 703{,}450/1{,}190 = 591\ 16/119 > 591\ 1/8$.

All of Archimedes' bounds to square roots could have been obtained in similar fashion, using side-and-diagonal numbers in essentially the way de Lagny suggested.

5. Heilermann. Heilermann [4], in 1881, suggested that the square root of any number n could be approximated by a series of side-and-diagonal numbers starting with

$$d_0/s_0 = 1/1 \tag{22}$$

and using the recursion formula

$$\frac{d_{t+1}}{s_{t+1}} = \frac{ns_t + d_t}{s_t + d_t}. \tag{23}$$

In other words, Heilermann let

$$v_0 = [s_0 \quad d_0] = [1 \quad 1] \quad \text{and}$$
$$T = \begin{bmatrix} 1 & n \\ 1 & 1 \end{bmatrix}. \tag{24}$$

Using (19) and (20) with $a=1$, we have for Heilermann's fractions $d_t^2/s_t^2 = n + (1-n)^t/s_t^2$. Thus Heilermann's d_t/s_t approaches $\sqrt{n}$ as t increases if $(1-n)^t/s_t^2$ approaches zero.

Heilermann used this procedure to get a single sequence of fractions to bound $\sqrt{3}$. The sequence is

$$\begin{aligned} &1/1,\ 4/2 = 2/1,\ 5/3,\ 14/8 = 7/4,\ 19/11,\ 52/30 = 26/15,\ 71/41,\ 194/112 \\ &= 97/56,\ 265/153,\ 724/418 = 362/209,\ 989/571,\ 2702/1560 \\ &= 1351/780, \cdots \end{aligned} \tag{25}$$

Note that whenever the numerator and denominator of a fraction contain a common factor, it is canceled before the next step. Thus, 4/2 is reduced to 2/1 before the next fraction is computed.

The even-numbered terms of Heilermann's sequence are identical to de Lagny's first sequence: 2/1, 7/4, 26/15, $\cdots$, while the odd-numbered terms are de Lagny's second sequence: 1/1, 5/3, 19/11, $\cdots$. The reason for this is that de Lagny's transformation matrix T_L is one-half the square of Heilermann's T_H. Thus

$$T_H = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad T_L = \frac{1}{2}\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}.$$

In this way, de Lagny's sequence skips every other term of Heilermann's.

Archimedes' bounds to $\sqrt{3}$ are the 9th and 12th terms of (25). Heilermann [4, pp. 122–123] recognized that Archimedes would probably not have used the rather loose bound 265/153 if he had computed the 12 terms to get the close bound 1351/780. Rather, he would have used 989/571. Therefore, Heilermann sought a modification of the method that would give Archimedes' bounds as two successive fractions in the same sequence.

To do this, he first found a sequence for $\sqrt{(27/25)} = 3\sqrt{3}/5$. Starting with

$$v_0 = [s_0 \quad d_0] = [1 \quad 1] \quad \text{and} \quad T = \begin{bmatrix} 1 & 27/25 \\ 1 & 1 \end{bmatrix},$$

he got the series 1/1, 26/25, 53/51, 1351/1300, $\cdots$. To approximate $\sqrt{3}$ any of these fractions would be multiplied by 5/3. Thus, the sequence for $\sqrt{3}$ would be

$$5/3,\ 26/25\cdot 5/3 = 26/15,\ 53/51\cdot 5/3 = 265/153,\ 1351/1300\cdot 5/3 = 1351/780,\ \cdots.$$

Eureka! Archimedes' two bounds are terms 3 and 4 of the same series. Heath [**3**, p. 97] wrote, "This is one of the very few instances of success in bringing out the two Archimedean approximations in immediate sequence without any foreign values intervening. No other methods appear to connect the two values in this direct way except those of Hunrath and Hultsch depending on the formula

$$a \pm b/2a > \sqrt{(a^2 \pm b)} > a \pm b/(2a \pm 1)."$$

Without wishing to detract in any way from the importance of Heilermann's excellent development, we note that the transformation

$$\frac{1}{2} T_H^3 = \frac{1}{2}\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}^3 = \begin{bmatrix} 5 & 9 \\ 3 & 5 \end{bmatrix}$$

provides a rapidly convergent sequence with desirable properties. Because its determinant is -2, the sequence oscillates about $\sqrt{3}$, as does the sequence generated by T_H; yet with the use of $T_H^3/2$ we get only every third term of Heilermann's sequence, so that convergence is rapid, and incidentally, Archimedes' bounds are adjacent in the sequence.

Starting with $v_0 = [3\ 5]$, we get directly the sequence for $\sqrt{3}$:

$$5/3,\ 52/30 = 26/15,\ 265/152,\ 2702/1560 = 1351/780,\ \cdots.$$

In general,

$$T_H^2 = \begin{bmatrix} 1 & n \\ 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1+n & 2n \\ 2 & 1+n \end{bmatrix}, \quad \text{and}$$

$$T_H^3 = \begin{bmatrix} 1 & n \\ 1 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1+3n & n(3+n) \\ 3+n & 1+3n \end{bmatrix}.$$

Whenever n is odd, each element of T_H^2 and of T_H^3 is divisible by 2, so we can use $T_H^2/2$ and $T_H^3/2$ as transformation matrices. This reduces the size of our fractions, and also reduces the error term. In fact, when d_t and s_t are divided by k, the error term is divided by k^2. For example, the first term of the above sequence, 5/3, gives an error $e_0 = 5^2 - 3\cdot 3^2 = -2$. If we use 52/30 as the second term we get $e_1 = 52^2 - 3\cdot 30^2 = +4$. But, if we use the equivalent fraction 26/15, we get $e_1 = 26^2 - 3\cdot 15^2 = +1$. If we use the reduced fractions, the errors are successively $-2, +1, -2, +1, \cdots$. If none of the fractions had been reduced, the errors would have been $-2, +4, -8, +16, \cdots$. Note, however, that the *relative* error, $e_t/s_t^2 = (d_t/s_t)^2 - n$, is unchanged by removal of the common factor.

6. Synthesis. The methods of Theon, de Lagny, and Heilermann are all of one generic kind. In general, let d_0/s_0 be an approximation to $\sqrt{n}$. The sequence of side-and-diagonal numbers is computed from $v_1 = Tv_0$, $v_2 = Tv_1 = T^2v_0, \cdots$, where $v_0 = [s_0 \;\; d_0]$ and

$$T = \begin{bmatrix} d_0 & ns_0 \\ s_0 & d_0 \end{bmatrix}. \tag{26}$$

Thus, the sequence of fractions is

$$\frac{d_0}{s_0}, \frac{d_1}{s_1} = \frac{ns_0^2 + d_0^2}{2s_0d_0}, \frac{d_2}{s_2} = \frac{ns_0s_1 + d_0d_1}{d_0s_1 + s_0d_1}, \cdots. \tag{27}$$

Using methods explained above, one can easily show that $d_{t+1}^2 - ns_{t+1}^2 = (d_0^2 - ns_0^2)(d_t^2 - ns_t^2) = (d_0^2 - ns_0^2)^t$. The system converges to $\sqrt{n}$ if $(d_0^2 - ns_0^2)^t/s_t^2$ approaches zero as t increases.

We can also get this sequence s_t, d_t by exponentiating $d_0 - s_0\sqrt{n}$, with $d_t - s_t\sqrt{n} = (d_0 - s_0\sqrt{n})^{t+1}$. If we choose s_0, d_0 so that $d_0 - s_0\sqrt{n} = k$, with $-1 < k < +1$, we have $d_t - s_t\sqrt{n} = k^{t+1}$, which approaches zero as t increases, because $|k| < 1$. Then certainly $d_t/s_t - \sqrt{n} = k^{t+1}/s_t$ approaches zero as t increases, since s_t increases, proving that d_t/s_t approaches $\sqrt{n}$.

From $d_{t+1} - s_{t+1}\sqrt{n} = (d_t - s_t\sqrt{n})(d_0 - s_0\sqrt{n})$, we have $d_{t+1} = d_0d_t + ns_0s_t$ and $s_{t+1} = s_0d_t + d_0s_t$, so that the exponentiation process accomplishes the same transformation as (26):

$$T = \begin{bmatrix} d_0 & ns_0 \\ s_0 & d_0 \end{bmatrix}.$$

Computing Archimedes' bounds to $\sqrt{3}$ by the system

$$v_0 = [3 \quad 5] \quad \text{and} \quad T = \begin{bmatrix} 5 & 9 \\ 3 & 5 \end{bmatrix}$$

can, then, be accomplished equivalently by exponentiating $5 - 3\sqrt{3}$:

$$(5 - 3\sqrt{3})^2 = 26 - 15\sqrt{3}, \; (5 - 3\sqrt{3})^3 = 265 - 153\sqrt{3},$$
$$(5 - 3\sqrt{3})^4 = 1351 - 780\sqrt{3}.$$

We know that $-1 < 5 - 3\sqrt{3} < +1$, so the sequence 5/3, 26/15, 265/153, 1351/780, $\cdots$ converges. We already knew this, of course, from a different standpoint.

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ON FINITE RINGS

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This note determines all finite rings whose additive group is a cyclic group. This result enables us to determine all finite rings, with a fixed additive group, having n elements, where n is a square-free integer and to determine when such rings are radical rings and when they are semisimple rings. We shall use $|S|$ to denote the cardinality of a finite set S. $\tau(n)$ represents the number of positive integral divisors of n. $\phi(n)$ is Euler's phi-function, that is, the number of positive integers less than n and relatively prime to n. Z indicates the ring of rational integers. Z_n is the ring of rational integers modulo n. (p, q) means the greatest common divisor of the integers p and q. Isomorphism means a one-to-one onto homomorphism.

It is clear that when the additive group of a finite ring R is a cyclic group generated by a nonzero element a, that is to say $\{a\}=R$, then the multiplicative structure of the ring is determined by a positive integer p such that $a^2=pa$. Henceforth, R_p represents a finite ring with $|R_p|=n$, whose additive group is cyclic and where $a^2=pa$. Let $\mathfrak{R}_p$ be the isomorphism class formed by all finite rings isomorphic to R_p. It is clear that if $\mathfrak{R}$ is the class of all finite rings whose additive group is a cyclic group of order n, $\mathfrak{R}$ is the disjoint union of the $\mathfrak{R}_p$.

Now let R_q belong to $\mathfrak{R}_p$. Then there exists an isomorphism $\alpha: R_q \to R_p$, given by $a\alpha = xa$, x in Z. By properties of congruence relations on Z_n, it is easy to show that $(x, n)=1$. This implies that the members of $\mathfrak{R}_p$ are given by considering all solutions y to the equation $px \equiv y \pmod{n}$, with $(x, n)=1$. Hence, if $d_1=(q, n)$ and $d_2=(p, n)$, then $d_1=qs+nt$ and $d_2=pr+nh$, for s, t, r, and h in Z; and since $px\equiv q \pmod{n}$, we have $q=px+nu$ for u in Z. Thus $d_1=pxs+nv$ and, therefore, $d_2 | d_1$. But if R_q belongs to $\mathfrak{R}_p$, then R_p belongs to $\mathfrak{R}_q$. Therefore, $qz\equiv p \pmod{n}$ and $p=qz+nw$ for w in Z. Hence, $d_2=qzr+ng$ for g in Z. Thus $d_1 | d_2$ and we have that $d_1=d_2$. Conversely, if $(p, n)=(q, n)=d$, then $q=q_1d$ and $d=pr+nt$ for r and t in Z. Thus $q=dq_1=prq_1+ntq_1$ and the congruence $px\equiv q$ $\pmod{n}$ has d incongruent solutions with at least one of them relatively prime to n. Therefore R_q belongs to $\mathfrak{R}_p$. We have proved

THEOREM 1. *R_q belongs to $\mathfrak{R}_p$ if and only if $(p, n)=(q, n)$.*

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THEOREM 1. *R_q belongs to $\mathcal{R}_p$ if and only if $(p, n)=(q, n)$.*

COROLLARY 1. *R_q belongs to $\mathcal{R}_p$ if and only if there exists x in Z such that $(x, n)=1$ and $px\equiv q \pmod n$.*

Let us remark that if $p|n$ and $q|n$, then R_q belongs to $\mathcal{R}_p$ if and only if $p=q$. This shows that all isomorphism classes $\mathcal{R}_p$ are given by the distinct divisors of n. Also, in order to determine the members of $\mathcal{R}_p$ it is enough to consider the divisors of n and multiply each of them by those integers that are less than n and relatively prime to n. Thus we have a proof of the following well-known result (see, for example [1] p. 263).

COROLLARY 2. *If n is a positive integer then there exist $\tau(n)$ nonisomorphic rings R, where $|R|=n$, and the additive group of R is cyclic. All such rings are commutative.*

COROLLARY 3. *If $(x, n)=1$, then $R_x\in\mathcal{R}_1$.*

Proof. $(x, n)=(1, n)$ and hence $R_x\in\mathcal{R}_1$.

COROLLARY 4. $|\mathcal{R}_1|=\phi(n)$, $|\mathcal{R}_0|=1$, *and if n is an even integer, then*

$$|\mathcal{R}_{n/2}| = 1.$$

COROLLARY 5. *All finite rings R with unit and $|R|=n$, whose additive group is cyclic, are isomorphic. They are commutative.*

Proof. If R has a unit element, R belongs to $\mathcal{R}_1$; since if R_p possesses a unit element, say $e=qa$ for some positive integer q, then $a=a(qa)=qa^2=qpa$ which implies that $qp\equiv 1 \pmod n$. Hence $(p, n)=1$ and, by Corollary 3, $R_p\in\mathcal{R}_1$.

Now let us determine the idempotent and nilpotent elements of all finite rings whose additive group is cyclic. An element e of Z_n is an idempotent element if and only if $e=cp$ or $e=bq$, where $n=pq$, $(p, q)=1$ and $1=cp+bq$ for c, b in Z [3]. If $n=\prod p_i^{t_i}$, $t_i\geqq 1$, $i=1 \cdots r$, p_i a prime integer, there are 2^r distinct idempotent elements in Z_n [3]. Let e be an idempotent element in R_p. Then $e=va$ for v a positive integer in Z and a the generator of the additive group of R_p. This implies that $v^2p\equiv v \pmod n$ and va is idempotent in R_p if and only if vp is idempotent in Z_{np}. Thus we have

PROPOSITION 1. *The idempotent elements of R_p are completely determined by the idempotent elements of Z_{np}.*

Let $n=\prod p_i^{t_i}$, $t_i\geqq 1$, $i=1, \cdots, r$, p_i an integer. An element b in Z_n is nilpotent if and only if $b=\prod p_i^{s_i}$, $s_i\geqq 1$, $i=1, \cdots, r$; that is b is nilpotent if and only if each p_i, $i=1, \cdots, s$, is a divisor of b, which implies that there are $\prod p_i^{t_i-1}$ $i=1, \cdots, s$ nilpotent elements in Z_n [3]. If b is a nilpotent element in Z_n, there exists a positive integer u such that $b^u=0 \pmod n$, but $b=ya$ for some y in Z. Thus $0=b^u=(ya)^u=y^up^{u-1}a$. Hence it is clear that b is a nilpotent element of R_p if and only if yp is a nilpotent element of Z_{np}. Thus we have

PROPOSITION 2. *The nilpotent elements of R_p are completely determined by the nilpotent elements of Z_{np}.*

Let us remark that if d is a nilpotent element of Z_n, R_d is a radical ring. R_p

is a semisimple ring (zero Jacobson radical) if and only if in Z_{np} there are no nilpotent elements of the form yp for y in Z.

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ANSWERS

A404. The $1/r_i$ are the roots of the reciprocal equation $3y^3-8y^2+6y-2=0$. Hence, by the relationships between the roots and coefficients, the given expression equals $6/2+6/3$ or 5. Also, the given expression is equivalent to $(r_1+r_2+r_3)(1+1/r_1r_2r_3)$, so is equal to $(6/2)\,[1+1/(3/2)]$ or 5.

A405. The proof is indirect. Assume two congruent and parallel chords of maximum length. The endpoints of these chords are the vertices of a parallelogram, one of whose diagonals, at least, is larger than all the sides. This contradicts our initial assumption and, consequently, we obtain our stated result.

A406. Using the comparison test for series, one finds that

$$\ln(n+1) = \int_1^{n+1}(1/x)dx < S_n.$$

So now we have

$$S_m - S_n > \ln(m+1) - \ln(n+1) = \ln\frac{m+1}{n+1}$$

and $\ln m+1/n+1>1$ whenever $m+1/n+1>e$. So by picking $m>(n-1)e-1$ we obtain the desired result. To simplify the choice of m, since $3>e$, the choice $m=3n+2$ will do.

A407.

$$\begin{aligned} e^{\ln 2} &= 2\exp\sum_{n=1}^{\infty}\frac{(\ln 2)^n}{n!} \\ &= 2\exp\sum_{n=0}^{\infty}\frac{(\ln 2)^n}{n!} - 1 \\ &= 2\exp e^{\ln 2} - 1 \\ &= 2. \end{aligned}$$

A408. Taking the rectangular solid with sides 1, 2, and 3, and volume 6, the lengths of the diagonals of its face are $\sqrt{5}$, $\sqrt{10}$, and $\sqrt{13}$. The tetrahedron can be formed by removing four corners of this rectangular body, cutting along diagonals. The volume of the tetrahedron is 2.

(Quickies on page 110)

THE TORSIONAL VIBRATION OF A SYSTEM OF DISKS ATTACHED TO A HEAVY SHAFT

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1. Introduction. The analysis of the undamped torsional vibrations of a heavy unloaded shaft and the analysis of the undamped torsional vibrations of a system of heavy disks attached to an elastic shaft of negligible moment of inertia are featured prominently in most texts on applied differential equations. Less well known, however, is the more realistic intermediate case in which the moment of inertia of the shaft as well as the moment of inertia of the disks attached to it is taken into account. There is one obvious qualitative difference between this problem and the idealized version of it in which the moment of inertia of the shaft is neglected: In the actual case, since the shaft is a continuous elastic body, the system has infinitely many natural frequencies with corresponding natural modes, while in the ideal case there is only a finite number of natural frequencies and normal modes. In practice it appears that the higher frequencies undisclosed by the lumped analysis are almost always outside the range of interest and that the frequencies and normal modes which are obtained are satisfactory approximations if the ratio of the moment of inertia of a typical disk to the moment of inertia of a typical segment of the shaft is even moderately large. One obvious, but very tedious, way to investigate these questions would be to solve the wave equation

$$\text{(1)} \qquad \frac{\partial^2\theta}{\partial t^2} = a^2\frac{\partial^2\theta}{\partial x^2}, \qquad a^2 = \frac{E_s g}{\rho}$$

for the angle of twist, $\theta(x, t)$, over each length of connecting shaft and then piece together these solutions so that θ is continuous at each disk and so that the torque, $E_sJ(\partial\theta/\partial x)$, has jumps equal to the inertia torque, $I(\partial^2\theta/\partial t^2)$, at each disk [1]. However (although this seems to have gone unnoticed except in the ideal case when the moment of inertia of the shaft is neglected) it is more convenient to focus attention on the disks rather than on the shaft, formulate the problem as one in ordinary rather than partial differential equations, and ultimately reduce it to the solution of a certain difference equation.

2. The natural frequencies. Consider a uniform shaft of length $L = nl$ bearing identical disks of moment of inertia I at the points $x = 0, l, 2l, \cdots, nl$. The relevant physical properties of the shaft are its modulus of elasticity in shear, E_s, its propagation velocity, a, and the polar moment of inertia of its cross sections, J. All frictional effects are assumed to be negligible, and no external torques act on the system.

We begin by asking the question: When a uniform shaft of length l vibrates torsionally at a single frequency, ω, in such a way that its ends move according to the laws

$$\text{(2)} \qquad \theta(0, t) = A_1 \sin \omega t, \qquad \theta(l,t) = A_2 \sin \omega t$$

what is the torque transmitted through its end sections? To answer this, we observe first that when the ends of the shaft move according to (2), then, from (1), the angle of twist at any point along the shaft is

$$(3) \qquad \theta(x, t) = \frac{1}{\sin\lambda}\left[A_1 \sin\lambda\left(1 - \frac{x}{l}\right) + A_2 \sin\lambda\frac{x}{l}\right]\sin\omega t, \qquad \lambda = \frac{\omega l}{a} \neq m\pi.$$

From this, since the torque transmitted through any cross section of a shaft is equal to $E_s J(\partial\theta/\partial x)$, it follows that the torques at the respective ends are

$$(4.1) \qquad T(0, t) = \frac{E_s J\lambda}{l}\left[\frac{-A_1 \cos\lambda + A_2}{\sin\lambda}\right]\sin\omega t$$

$$\lambda \neq m\pi.$$

$$(4.2) \qquad T(l, t) = \frac{E_s J\lambda}{l}\left[\frac{-A_1 + A_2 \cos\lambda}{\sin\lambda}\right]\sin\omega t$$

These expressions are generalizations, to the dynamic case, of the familiar Hooke's law formulas

$$T(0, t) = -T(l, t) = \frac{E_s J}{l}(-\theta_1 + \theta_2)$$

which apply to the static twisting of a shaft (or to the dynamic twisting of a shaft when its moment of inertia is neglected).

Now in the system we are considering, since friction and all external torques are neglected, each disk moves solely by virtue of the torques transmitted to it by the adjacent lengths of shafting. Hence, using Newton's Law together with (4.1) and (4.2) to formulate the differential equations for the angular displacements, $\theta_k = A_k \sin\omega t$, of the respective terms, then dividing out $\sin\omega t$ and collecting terms, we obtain the algebraic system

$$(5.1) \qquad (\cos\lambda - r\lambda\sin\lambda)A_1 - A_2 = 0$$

$$(5.2) \qquad A_{k-1} - (2\cos\lambda - r\lambda\sin\lambda)A_k + A_{k+1} = 0 \qquad k = 2, 3, \cdots, n-1$$

$$(5.3) \qquad -A_{n-1} + (\cos\lambda - r\lambda\sin\lambda)A_n = 0$$

where

$$r = \frac{\text{moment of inertia of one disk}}{\text{moment of inertia of one length of shaft}}.$$

Equation (5.2) is a simple difference equation whose solution, subject to the boundary conditions (5.1) and (5.3), leads to a complete description of the free motion of the system.

In solving (5.2) there are five cases to consider,

(a) $\qquad 2\cos\lambda - r\lambda\sin\lambda < -2$

(b) $\qquad 2\cos\lambda - r\lambda\sin\lambda = -2$

(c) $\qquad -2 < 2\cos\lambda - r\lambda\sin\lambda < 2$

(d) $\qquad 2\cos\lambda - r\lambda\sin\lambda = 2$

(e) $\qquad 2\cos\lambda - r\lambda\sin\lambda > 2$

In (a) and (e) it is easy to show that the system (5) has only the trivial solution $A_k = 0$, $k = 1, 2, \cdots, n$. In (b) it turns out that

$$\lambda = (2N-1)\pi \quad \text{and} \quad A_k = (-1)^k C \qquad (C \text{ arbitrary})$$

while in (d) it turns out that

$$\lambda = 2N\pi \quad \text{and} \quad A_k = C \qquad (C \text{ arbitrary}).$$

Hence neither of these is a proper case of (5) since (5) was obtained on the assumption that $\lambda \neq m\pi$. Checking these independently, however, by taking ω to be $(2N-1)a\pi/l$ and then $2Na\pi/l$ in (2) and retracing the steps that led to (5), it follows readily that in these cases, also, only the trivial solution is possible.

Finally, when $-2 < 2 \cos \lambda - r\lambda \sin \lambda < 2$ we can write

$$2 \cos \mu = 2 \cos \lambda - r\lambda \sin \lambda. \tag{6}$$

Hence the solution of (5.2) becomes $A_k = c_1 \cos k\mu + c_2 \sin k\mu$. Imposing the boundary conditions (5.1) and (5.3) on this expression leads to two homogeneous linear equations in c_1 and c_2 which will have a nontrivial solution if and only if

$$\cos^2 \lambda \sin (n-1)\mu - 2 \cos \lambda \sin n\mu + \sin (n+1)\mu = 0. \tag{7}$$

Equation (7) can be solved rationally for $\cos \lambda$, and we have for all values of n the two possibilities

$$\cos \lambda = \frac{\sin \frac{n+1}{2} \mu}{\sin \frac{n-1}{2} \mu} \tag{8.1}$$

$$\cos \lambda = \frac{\cos \frac{n+1}{2} \mu}{\cos \frac{n-1}{2} \mu}. \tag{8.2}$$

These, with (6), define the frequency parameter $\lambda = \omega l/a$ as a function of the inertia ratio r.

As a specific example, let $n = 4$. Then, eliminating μ between (6), (8.1) and (8.2), we obtain in the respective cases

$$r = \frac{3 \cos \lambda + 1 \pm \sqrt{\cos^2 \lambda + 2 \cos \lambda + 5}}{2 \lambda \sin \lambda} \tag{9.1}$$

$$r = \frac{3 \cos \lambda - 1 \pm \sqrt{\cos^2 \lambda - 2 \cos \lambda + 5}}{2 \lambda \sin \lambda}. \tag{9.2}$$

From these the (r, λ)-plots shown in Fig. 1 can be constructed at once. To compare these curves with the corresponding ones when the moment of inertia

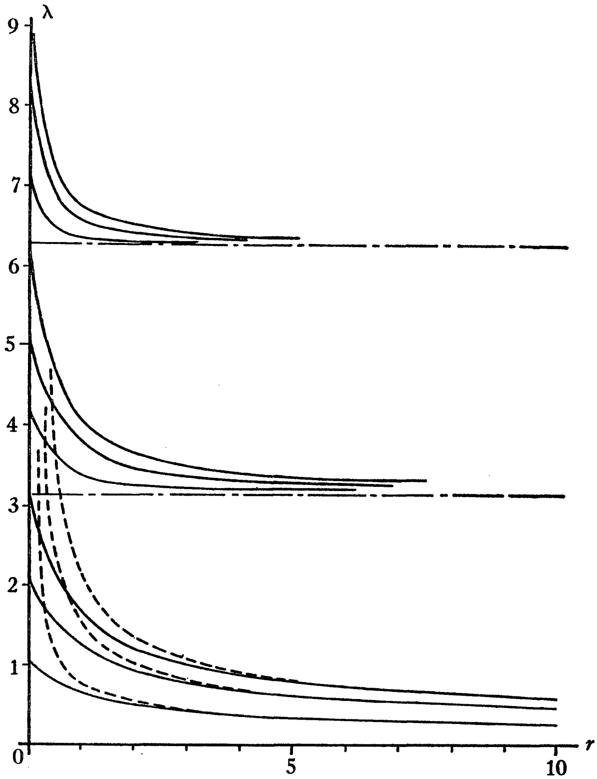

FIG. 1.

of the shaft is neglected, we recall that in the ideal case [2],

$$\omega = 2\sqrt{\frac{E_s J}{lI}} \sin\frac{k\pi}{2n} \qquad k = 1, 2, \cdots, n-1.$$

Hence

$$\lambda = \frac{\omega l}{a} = \frac{2}{\sqrt{r}} \sin\frac{k\pi}{2n}\,\cdot$$

Plots of this relation for $n=4$ and $k=1, 2, 3$ appear as dotted curves in Fig. 1. Clearly, over a wide range of inertia ratios the first three natural frequencies in the actual case are almost identical with the respective frequencies in the ideal case.

One unusual property of the system we have been considering is suggested by Fig. 1. Whereas in most boundary value problems the characteristic values are asymptotically equally spaced, this is not the case in the present problem. In fact, from Fig. 1 it appears that within successive groups of three the interval between consecutive characteristic values approaches zero, while the interval between characteristic values in successive groups approaches π.

That this is indeed the case can be established by considering (9.1) and (9.2). From these equations it is clear that the characteristic values are the abscissae of the intersections of the curve

$$y = 2rx \sin x \tag{10}$$

and the curves

$$y = 3 \cos x + 1 \pm \sqrt{(\cos^2 x + 2 \cos x + 5)} \tag{11.1}$$

$$y = 3 \cos x - 1 \pm \sqrt{(\cos^2 x - 2 \cos x + 5)}. \tag{11.2}$$

The curve defined by (10) has intercepts at $x = m\pi$, between which its ordinates become unbounded as m becomes infinite and at which it crosses the x-axis in a direction which approaches the vertical as m becomes infinite. The four curves defined by (11.1) and (11.2) are bounded periodic curves of period 2π. Moreover, the curve defined by (11.1) when the plus sign is taken crosses the x-axis where $x = (2N-1)\pi$, and the curve defined by (11.2) when the minus sign is taken crosses the x-axis where $x = 2N\pi$. Hence, neglecting the irrelevant intersections at $x = m\pi$ on one or the other of these curves, it is clear that as m becomes infinite the intersections of the first curve and the other three fall, in successive groups of three, more and more nearly on the line $x = m\pi$, as asserted.

3. The normal modes. In the ideal case when the moment of inertia of the shaft is neglected, the normal mode corresponding to the k-th natural frequency is simply an ordered set of n numbers describing the relative amplitudes through which the respective disks oscillate when the system is vibrating with frequency ω_k. On the other hand, in the actual case the normal modes are continuous functions of x whose derivatives are discontinuous at $x = l, 2l, \cdots, (n-1)l$. These can easily be found from (3) once the difference equation (5.2) has been solved for the A's.

In the ideal case it is well known that two distinct normal modes, say $(A_1, A_2, \cdots, A_n)$ and $(B_1, B_2, \cdots, B_n)$ satisfy the orthogonality relation $\sum_{k=1}^{n} A_k B_k = 0$. It is also well known that any two distinct normal modes of a uniform unloaded shaft, say $f(x)$ and $g(x)$, satisfy the orthogonality relation $\int_0^l f(x)g(x)dx = 0$. In the actual case of a heavy shaft loaded with a series of disks, neither of these orthogonality conditions holds and, as expected, we have instead a condition which is a combination of them.

Specifically, if the product of two distinct normal modes, say $f(x)$ and $g(x)$, as given by (3) is integrated over the length of the shaft and the result simplified by means of (4), we obtain the generalized orthogonality condition

$$I_s \int_0^l f(x)g(x)dx + I \sum_{k=1}^{n} A_k B_k = 0 \tag{12}$$

where I_s is the moment of inertia of the shaft per unit length. It is interesting, and not unexpected, that the two terms in (12) can be combined into a single integral of the Stieltjes type

$$\int_0^l f(x)g(x)d\sigma(x) \tag{13}$$

where the integrator function $\sigma(x)$ is the (discontinuous) function which gives the total moment of inertia of the system between the left end of the configuration and the general point x. In terms of Stieltjes integration, the expansion of arbitrary initial conditions of angular displacement and velocity can be carried out formally in a manner identical with the familiar Riemann integral treatment of the corresponding problem for unloaded shafts.

4. Conclusion. Although this note has been concerned only with the analysis of a series of identical disks coupled by identical lengths of uniform shafting, the method we have presented is applicable to a much wider class of cases. Obviously, our method can be applied equally well to configurations with dissimilar disks at each end and to configurations one or both of whose ends are fixed rather than free.

More generally, our method can be applied without essential change to systems in which adjacent disks are coupled not by identical lengths of shafting but by identical configurations of any kind, for instance by uniform shafts of length l each bearing at its midpoint a disk of moment of inertia $I_1 \neq I$. It is only necessary to compute the displacement function analogous to (3) and from it the torque expressions analogous to (4.1) and (4.2) for the given coupling configuration, and then proceed in precisely the way we have described.

References

1. S. Timoshenko, Vibration Problems in Engineering, D. Van Nostrand, New York, 1937, p. 326.

2. T. v. Karman and M. A. Biot, Mathematical Methods in Engineering, McGraw-Hill, New York, 1940, p. 459.

ALGORITHMS THAT USE TWO NUMBER SYSTEMS SIMULTANEOUSLY

EDGAR KARST, University of Arizona

Before the year 1955 a title as the one above would have been considered paradoxical if not absurd. But the development of program-controlled computers, especially of those with nondecimal in- and output, made the construction of unusual algorithms feasible.

Our goal may be a very practical one: to find a multiplication algorithm for nondecimal systems with as little conversion as possible. Under conversion we mean: conversion from one system to another and back. Without loss of generality we may assume positive integral factors.

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The usual method, valid for all systems with positive integral bases >1, may be expressed as: Write the remainder down and use the multiple of the base as carry. For example, in base 8:

$$\begin{array}{r} 4567_8 \\ 123 \\ \hline 16145 \\ 11356 \\ 4567 \\ \hline 610625_8 \end{array}$$

But this method becomes inconvenient for bases >10. For example, in base 16:

$$\begin{array}{r} DEF_{16} \\ ABC \\ \hline A734 \\ 9945 \\ 8B56 \\ \hline 959184_{16} \end{array}$$

Hence we use the dibble-dabble method as a checking device:

$$DEF_{16} = D\cdot 16^2 + E\cdot 16 + F = 13\cdot 256 + 14\cdot 16 + 15 = 3567_{10}$$
$$ABC_{16} = A\cdot 16^2 + B\cdot 16 + C = 10\cdot 256 + 11\cdot 16 + 12 = 2748_{10}$$

$$\begin{array}{r} 3567_{10} \\ 2748 \\ \hline 28536 \\ 14268 \\ 24969 \\ 7134 \\ \hline 9802116_{10} \end{array} \qquad \begin{array}{rcrcl} 9802116 & : & 16 = & 612632 & R\ \ 4 \uparrow \\ 612632 & : & 16 = & 38289 & R\ \ 8 \\ 38289 & : & 16 = & 2393 & R\ \ 1 \\ 2393 & : & 16 = & 149 & R\ \ 9 \\ 149 & : & 16 = & 9 & R\ \ 5 \\ 9 & : & 16 = & 0 & R\ \ 9 \end{array}$$

If the remainders R are read from bottom to top, we obtain the product above in base 16.

Though the two bases were chosen arbitrarily, there exist program-controlled computers accepting only numbers in those bases for in- and output. They have one in common: $8=2^3$ and $16=2^4$. Hence the conversion from one base, 2^m, to another, 2^n, can be done digit-wise, for example:

Bases and Equivalents of their Digits

16	8	4	2	16	8	4	2
0	0	00	000	8	10	20	1000
1	1	01	001	9	11	21	1001
2	2	02	010	*A*	12	22	1010
3	3	03	011	*B*	13	23	1011
4	4	10	100	*C*	14	30	1100
5	5	11	101	*D*	15	31	1101
6	6	12	110	*E*	16	32	1110
7	7	13	111	*F*	17	33	1111

If, therefore, a convenient algorithm can be found for base 2, it will be applicable also for bases of powers of 2. Unfortunately, the usual method is too long as we may see from the first two examples, which we have converted to base 2, digit-wise:

4567_8
123
100101110111_2
1010011
100101110111
100101110111
100101110111
100101110111
110001000110010101_2

6 1 0 6 2 5_8

DEF_{16}
ABC
110111101111_2
101010111100
00
110111101111
110111101111
110111101111
110111101111
110111101111
110111101111
110111101111
$100101011001000110000100_2$

9 5 9 1 8 4_{16}

As the interested reader may easily recognize, it becomes cumbersome to keep track of the carries in adding the many units and in computing the final digit in each column, especially in the middle ones. Hence a few clever computing enthusiasts came up with the question: Can somebody concoct an algorithm by means of which we would be able to calculate almost entirely in the decimal mode with a little conversion at the end and maybe at the start? In other words: we would like to have algorithms that use two number systems simultaneously, with emphasis on the decimal system, if possible.

PARSONS' ALGORITHM

```
             4567_8
              123
       ------------
       100101110111_2
            1010011
       ------------
       100101110111
      100101110111
    100101110111
  100101110111
  ------------------
  101103222324321221
      888888888 88
     1211121320210
      8    8 88 8
     20   2022010
     8    8 88
    20   20110
    8    8
   20   20
   8    8
  10   10
  ------------------
  110001000110010101_2

   6  1  0  6  2  5_8
```

```
                                DEF_16
                                ABC
                       -------------
                         110111101111_2
                         101010111100
                       -------------
                                   00
                       110111101111
                      110111101111
                     110111101111
                    110111101111
                  110111101111
                110111101111
              110111101111
              -----------------------
              11122334556565443432100
                 8888888888888888
                211223445454332320
                8  88888888888888
               20 211233434322120
               8  8   888888888 8
              20 20 212232321020
              8  8  8 888888  8
             10 10 202112120 10
                   8 8  8 8
                 1010 2020
                     8 8
                    2010
                     8
                    10
             ------------------------
             100101011001000110000100_2

              9   5   9   1   8   4_16
```

The answer was not easy to obtain. In fact, there was ample doubt (for example: the lack of any clue in Dickson's *History of the Theory of Numbers*) that such algorithms could be found. Hence, the year 1955 in which F. L. Parsons [2] published *A Simple Desk-Calculator Method for Checking Binary Results* can be considered the year of birth of the first algorithm that uses two number systems simultaneously. (See page 94.)

Except for the last two lines in the former two examples, Parsons' Algorithm has the same start. But instead of adding in the binary mode

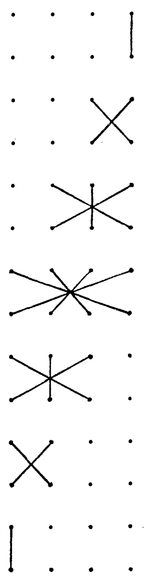
(1) we add in the decimal mode,
(2) we write below each digit other than 0 or 1 (that is below each digit which is not in the system) the 10-complement of the base,
(3) we keep adding (always the last two entries in each column) in the decimal mode,
(4) we transfer the bottom digit of each column to the row of the result.

It should be mentioned that the calculations with the 10-complement are easily done by means of a desk calculator, since we have to strike only the keys 8 and +, starting from right to left.

After an algorithm of this kind was known, it was a comparatively simple task to reduce the tedious intermediate computations by using Bhaskara's one-digit cross-multiplication method for the decimal system, an algorithm which employs no intermediate calculation at all.

Let a $(=4567)$ in digit length be greater or equal to b $(=123)$, and let ab $(=561741)$ be wanted without intermediate calculation. If a has n $(=4)$ digits, then Bhaskara's Algorithm consists of the following $2n-1$ $(=7)$ steps:

BHASKARA'S ALGORITHM



$= 3\cdot 7 = 21$

$= 2 + 3\cdot 6 + 2\cdot 7 = 34$

$= 3 + 3\cdot 5 + 2\cdot 6 + 1\cdot 7 = 37$

$= 3 + 3\cdot 4 + 2\cdot 5 + 1\cdot 6 + 0\cdot 7 = 31$

$= 3 + 2\cdot 4 + 1\cdot 5 + 0\cdot 6 = 16$

$= 1 + 1\cdot 4 + 0\cdot 5 = 05$

$= 0 + 0\cdot 4 = 00$

Since the result is developed digit-wise from right to left, it should be possible to obtain by some short kind of manipulation with the 10-complement of the base the corresponding octal result, i.e., $4567_8 \cdot 123 = 610625_8$.

Here, the author's investigations began in 1956 [1]:

KARST'S ALGORITHM

4567_8	*a*
123	*b*
561741	*c*
46884	*d*
610625_8	*e*

In the first three rows (*a*, *b*, *c*) we recognize Bhaskara's Algorithm, but we develop rows (*c*, *d*, *e*) almost simultaneously. As soon as we have $3 \cdot 7 = 21$, we have also

(1) the rightmost digit of row *c*,
(2) the decimal carry 2,
(3) the multiple 2 of $2 \cdot 8 + 5 = 21$,
(4) the product of multiple times 10-complement ($2 \cdot 2 = 4$).

This 4 becomes the rightmost digit of row *d* which is added to the rightmost digit of row *c* in the decimal mode, yielding the rightmost digit of row *e* which is also the rightmost digit of the wanted octal result. In a similar manner we develop the second digit-column from right. From the second step in Bhaskara's Algorithm, $2 + 3 \cdot 6 + 2 \cdot 7 = 34$, we have the 4 of row *c* and the carry 3. Also we have $4 \cdot 8 + 2 = 34$ and $4 \cdot 2 = 8$ which becomes the second digit from right in row *d*. Adding $4 + 8$ in the decimal mode we obtain 2 in row *e* with another carry 1. Hence, if we do this in our head we have to keep track of two carries. Should adding of any digit column in (*c*, *d*) inclusive carry result in a digit which is not in the nondecimal system, only this digit (in the result row) has to be converted. For example, carrying $1 + 1 + 6$ in the 4th column from right would yield 8; but $8_{10} = 10_8$. Therefore, row *e* shows a 0 with carry 1.

After this algorithm became widely applied as a checking device, a few prime number enthusiasts asked: Can someone concoct a simple multiplication algorithm for the base 6, since prime numbers (except 2 and 3) are of the form $6m \pm 1$? Or more generally: Can we obtain an algorithm which will work in bases 2 up to 10? Here we discover that the author's algorithm may be freely extended from base 2 up to base 10 (base 10 is included, since adding the 10-complement 0 would make row *c* equal to row *e* while row *d* would become zero). The following examples are given to satisfy the interested reader's curiosity; they are conversions of $56789_{10} \cdot 1234$ into the equivalents in bases 9 down to 6.

KARST'S ALGORITHM

85808_9	a	156725_8	a	324365_7	a	1114525_6	a
1621	b	2322	b	3412	b	5414	b
139094768	c	363915450	c	1106733380	c	6034038350	c
16673610	d	49330822	d	370702263	d	4507962892	d
155768378_9	e	413246272_8	e	1510435643_7	e	10542001242_6	e

References

1. Edgar Karst, A simple octal (binary) multiplication method for checking computer results (review), Math. Tables Aids Comput., 10 (1956) 100.

2. F. L. Parsons, A simple desk calculator method for checking binary results of digital-computer arithmetic operations, J. Assoc. Comput. Mach., July (1955).

BOOK REVIEWS

EDITED BY DMITRI THORO, San Jose State College

Materials intended for review should be sent to: Dmitri Thoro, Department of Mathematics, San Jose State College, San Jose, California 95114.

Ways of Thought of Great Mathematicians. By Herbert Meschkowski. Translated by J. Dyer-Bennet. Holden-Day, San Francisco, 1964. viii+110 pp. $2.95 (paper).

When an author wishes to illustrate 2500 years of mathematical thought in nine brief chapters, his choice of topics is necessarily arbitrary. Professor Meschkowski has selected some of the giants (Archimedes, Gauss), and some whose contributions are not as well known (Nicholas of Cusa, George Boole). In each chapter he places the man in his era of history, then presents one or two of his contributions to mathematics. The topics covered are these: (1) Pythagoras and the Pythagoreans—Pythagorean numbers and the Golden Section; (2) Archimedes—a proof of the equation for the surface area of a sphere; (3) Nicholas of Cusa—squaring the circle; (4) Pascal—mathematical induction applied to Pascal's triangle, rules for proofs in geometry; (5) Leibniz—the "harmonic triangle," Leibniz's series for $\pi/4$, and an exchange of letters between Leibniz and Varignon concerning the "infinitely small"; (6) Gauss—an analytic proof of the fundamental theorem of algebra; (7) Boole—Boolean algebra and an application to probability; (8) Weierstrass—the arithmetization of analysis and a letter from H. A. Schwarz to Georg Cantor indicating Weierstrass's working methods; and (9) Cantor—set theory, a letter from Cantor to F. Goldscheider introducing Cantor's investigations, and an example of an uncountable set. In each case the author presents material chosen to illustrate the way of thought of the man involved.

Karst's Algorithm

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1621	b	2322	b	3412	b	5414	b
139094768	c	363915450	c	1106733380	c	6034038350	c
16673610	d	49330822	d	370702263	d	4507962892	d
155768378_9	e	413246272_8	e	1510435643_7	e	10542001242_6	e

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The mathematics presented in the first four chapters is accessible to anyone with a knowledge of precalculus math; in the last five chapters the author presents more advanced topics, which still can be read with profit by one willing to follow the reasoning given. The translation is smooth. Professor Meschkowski's style is always clear, often informal, and occasionally humorous, as when he cautions the reader accustomed to the rigor of modern analysis to, "quiet his mathematical conscience with a fairly powerful sleeping pill, and then later try to translate this argument into the language of modern calculus." There are few typographical errors; most of those found were in the chapter on George Boole.

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Katz presents a worthwhile, self-contained, undergraduate text, with only high school algebra as prerequisite. It provides the foundation for calculus and a background for modern algebra based on the axiomatic development of the real number system conceived as a complete ordered field. The seventeen axioms concern a set, denoted by the symbol R, and the addition, multiplication and max functions. Topics include extrema of number sets, inductive proof, completeness, irrational powers, real functions, $\sum$, and the Bernoulli inequalities for real powers. Sixty-five lessons in three relatively independent parts provide the careful reader with good mathematical introductions to logic (thirteen lessons), sets, ordered pairs, and functions (eight lessons) as preliminaries to the real number system (forty-four lessons). The author's treatment of logic rests on the idea of similar statements and his development of sets is limited by the use of only two axioms. Many examples and notes are used throughout the book to strengthen the step-by-step development; to illustrate the consequences and uses of definitions, axioms, theorems and notation; to clarify ideas and point out the usual pitfalls a student encounters in his first venture into the mathematical world of axiomatics. The division of each lesson into numbered sections with numerous sectional references, the over 600 exercises essential to development, the maximum use of symbolism, and the briefness of exposition limits the usefulness of this book as a "quick" reference. A "Glossary of Special Symbols" and an index (both with awkward references to numbered sections), and a selected bibliography are provided.

E. A. WIXSON, Plymouth State College

Elementary Probability. By Edward O. Thorp. Wiley, New York, 1966. vii+152 pp. $4.95.

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Chapter 1 provides the student with insight into the importance of probability as a tool for the physical sciences and its historical genesis as a tool to analyze gambling. Combinatorial problems are then considered along with finite induction and summation. Chapter 2 consists of set theory, probability on finite sample spaces, functions, conditional probability, Bayes' Rule and independence. Chapter 3 covers countable sample space, random variables, and expectations. Chapter 4 is concerned with continuous probability.

Some of the highlights of the treatment are numerous pertinent examples to motivate and illuminate the theoretical development and an adequate number of interesting problems which reinforce and extend the theoretical development. Also, a number of concepts which are cornerstones of advanced probability theory are carefully and clearly developed in this book.

In short, Thorp's book sets a fine example for what an elementary book in mathematics should be.

R. S. BUCY, Univ. of Southern California

Elements of the Theory of Probability. By Emile Borel. Translated by John E. Freund. Prentice-Hall, Englewood Cliffs, New Jersey, 1965. xii+178 pp. $5.75.

This book of 178 pages is divided into two parts and four appendices. Part I, consisting of 65 pages, is devoted to discrete probability. In Part II of 83 pages, continuous probabilities are discussed.

In chapters 1 through 6, the following material is presented: The binomial distribution is introduced, and the concept of mathematical expectation is defined. Four solved problems illustrate the concepts. Next, the theorem of total probabilities (for pairwise disjoint events) and the multiplicative theorem are proved. Eight solved problems serve as illustrations. Chapter 3 is devoted mainly to Stirling's formula. Three approximations are given. In what follows, the (fair) coin tossing problem is taken up, and the local and integral limit theorems are derived. Then the results are generalized to the case of a not necessarily fair coin. The (weak) law of large numbers is proved and numerical examples are given. In these last two chapters the terminology being used in connection with the normal distribution differs from standard terminology. Here the discussion of discrete probabilities is concluded.

In the first three chapters of Part II, geometrical probabilities are introduced on sets in the real line, in the plane, and in three dimensional space. Contradictory answers arising from various definitions of geometrical probabilities are presented by means of classical examples. The study of geometrical probabilities continues throughout the solutions of ten interesting problems. Applications of the geometric probabilities to the distribution of minor planets and gases are then presented.

In discussing errors of measurements, the author again uses terminology different from the standard. In the following four chapters, the Poisson Distribution is derived as the limit of the binomial distribution and the discrete and continuousversionsof the Bayes Theorem are introduced. This theorem is illustrated by a number of examples. In the final chapter, some applications of the theory

Chapter 1 provides the student with insight into the importance of probability as a tool for the physical sciences and its historical genesis as a tool to analyze gambling. Combinatorial problems are then considered along with finite induction and summation. Chapter 2 consists of set theory, probability on finite sample spaces, functions, conditional probability, Bayes' Rule and independence. Chapter 3 covers countable sample space, random variables, and expectations. Chapter 4 is concerned with continuous probability.

Some of the highlights of the treatment are numerous pertinent examples to motivate and illuminate the theoretical development and an adequate number of interesting problems which reinforce and extend the theoretical development. Also, a number of concepts which are cornerstones of advanced probability theory are carefully and clearly developed in this book.

In short, Thorp's book sets a fine example for what an elementary book in mathematics should be.

R. S. BUCY, Univ. of Southern California

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There are a few misprints to be noted. In the footnote of page 26, one may add "$|x| < 1$." In line 11 of page 69, the second "i" in the word "mathematician" is missing. Also in the footnote, the word "not" should be inserted after "we" in the second line. In line 1, page 70, the word "hence" should be deleted and "unacceptable" should replace "inacceptible" in line 22. In the first line of page 125 "w_0" should precede "w_1." On page 135, line 18, "$BG+GB$" rather than "BG" should be written in accordance with line 28, page 134.

In appendix 1, the author discusses what he terms psychological games, and also the impossibility of imitating chance. Appendix 2 is devoted to objective and subjective probabilities and, finally, the Petersburg Paradox is presented in appendix 3.

The inadequate amount of material presented and the out of date mathematics which is used do not qualify this book as a textbook for a course in modern probability theory. However, the philosophical approach of probability, which makes itself evident throughout the book, and a number of interesting examples not usually found in texts, make this book highly recommendable for supplementary reading.

Professor Freund is to be thanked for making this book available to English speaking readers.

GEORGE ROUSSAS, University of Wisconsin

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in india ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles, California 90029.

PROBLEMS

649. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Solve the cryptarithm

$$\begin{array}{r} THREE \\ +\quad FOUR \\ \hline SEVEN \end{array}$$

in the decimal system such that:

of probability (distribution of stars, atomic weight, biometrical applications) are given.

There are a few misprints to be noted. In the footnote of page 26, one may add "$|x| < 1$." In line 11 of page 69, the second "i" in the word "mathematician" is missing. Also in the footnote, the word "not" should be inserted after "we" in the second line. In line 1, page 70, the word "hence" should be deleted and "unacceptable" should replace "inacceptible" in line 22. In the first line of page 125 "w_0" should precede "w_1." On page 135, line 18, "$BG+GB$" rather than "BG" should be written in accordance with line 28, page 134.

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Solve the cryptarithm

$$\begin{array}{r} THREE \\ +\quad FOUR \\ \hline SEVEN \end{array}$$

in the decimal system such that:

3 does not divide $T\ H\ R\ E\ E$ in which the digit 3 is missing;
4 does not divide $F\ O\ U\ R$ in which the digit 4 is missing;
7 does not divide $S\ E\ V\ E\ N$ in which the digit 7 is missing.

650. *Proposed by Charles W. Trigg, San Diego, California.*

At a distance from each side of the triangle ABC equal to the length of that side and on the vertex side of that side, a line is drawn parallel to that side. These three lines determine a triangle $A'B'C'$ similar to ABC. Show that $A'A$, $B'B$ and $C'C$ are concurrent at a point P whose distances from the sides of ABC are proportional to the sides.

651. *Proposed by Frank Dapkus, Seton Hall University.*

Let $x^2+bx+c=0$ have roots x_0 such that $|x_0| \leqq a$. What is the probability that x_0 is real?

652. *Proposed by Merrill Barnebey, Wisconsin State University at LaCrosse.*

Prove that the sum of any pair of twin primes greater than seven is divisible by twelve.

653. *Proposed by Sam Newman, Atlantic City, New Jersey.*

What is dy/dx of

$$y = \underbrace{x^{x^{x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}}}}_{n} \qquad ?$$

654. *Proposed by Sidney Kravitz, Dover, New Jersey.*

Ten kids own a total of 2,879 pennies. The ratio of every kid's fortune to the fortune of each kid poorer than himself is an integer. If no two kids own the same amount, find the fortune of each kid.

655. *Proposed by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan.*

It is easy to show that any two spheres are homothetic, regardless of their orientation. Show that this property characterizes spheres; that is, if two bounded figures are homothetic, regardless of their orientation, then they both must be spheres.

SOLUTIONS

Late Solutions

Kenneth A. Ribet, Brown University: **615**; *L. J. Upton, Port Credit, Ontario, Canada*: **613**.

Pythagorean Alphametic

628. [September, 1966] *Proposed by B. Suer and Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Solve the alphametic,

$$COS^2 + SIN^2 = UNO^2$$

in the decimal system.

Solution by J. A. H. Hunter, Toronto, Ontario, Canada.

We have $S^2+N^2 \equiv O^2$ (mod 10), and obviously $S \neq$ zero. For each N, for $N=0, 1, \cdots, 9$, we tabulate possible S and corresponding O values, bearing in mind digital "square-endings."

Since $U>S$, we then test each possible UNO value to find its representations (if any) as sum of two squares: bearing in mind the conditions which are required for this to be possible. Where representation as sum of squares is possible, we can then note corresponding SIN and COS from the well-known solution:

$$(x^2 + y^2)^2k^2 = (x^2 - y^2)^2k^2 + (2xy)^2k^2.$$

The working is somewhat tedious, but not unduly so. It is found that uniquely we have

$$391^2 + 120^2 = 409^2.$$

Also solved by R. H. Anglin, Danville, Virginia; Merrill Barnebey, Wisconsin State University at LaCrosse; Sarah Brooks, Utica Free Academy, New York; Jack Dix, Rutgers University; Charles R. Fleenor, Ball State University, Indiana; Michael Goldberg, Washington, D. C.; Jerome J. Schneider, Chicago, Illinois; Charles W. Trigg, San Diego, California; and the proposers.

A Sliding Locus

629. [September, 1966] *Proposed by C. Stanley Ogilvy and Stephen Barr, Hamilton College, New York.*

Rectangle $OPQR$ is initially placed so that OP lies along the positive x-axis and OR lies along the positive y-axis. If the rectangle is rotated through 90° in such a way that O slides along the x-axis and R slides along the y-axis, what is the locus of Q?

Solution by Sister M. Stephanie Sloyan, Georgian Court College.

Let $OP=b$ and $PQ=a$, and place the rectangle in some intermediate position with O on the positive x-axis and R on the positive y-axis so that the positive x-axis makes with the line OP an angle θ. Let coordinates of Q be (x, y). Then $x=b \cos \theta$ and $y=a \cos \theta+b \sin \theta$. As θ varies from 0° to 90°, the locus of Q is a portion of the ellipse

$$\frac{x^2(a^2 + b^2)}{b^2} - \frac{2axy}{b} + y^2 = b^2.$$

This portion is such that $0 \leqq x \leqq b$ and $a \leqq y \leqq \sqrt{(a^2+{}^2b)}$.

However, if the rectangle is allowed to be moved about in the four quadrants, always keeping R on the y-axis and O in the x-axis, the entire ellipse can be obtained as the locus of Q.

Also solved by P. N. Bajaj, Western Reserve University; Merrill Barnebey, Wisconsin State University at LaCrosse; Wray G. Brady, University of Bridgeport, Connecticut; R. J. Cormier, Northern Illinois University; Jack Dix, Rutgers University; Tadashi Fujiwara, Hyogo-Ken, Japan; Michael Goldberg, Washington, D. C.; Richard A. Jacobson, Houghton College, New York; Douglas Lind, University of Virginia; Maurice Nadler, Pace College, New York; Paul Sugarman, Swampscott High School, Massachusetts; John W. Warren, Odessa, Texas; and the proposers. Two incorrect solutions were received.

A Continuous Inverse Function

630. [September, 1966] *Proposed by C. J. Mozzochi, University of Connecticut.*

Prove: If f is real valued, strictly monotone increasing, and defined almost everywhere on $[a, b]$, then f^{-1} is continuous everywhere on its domain.

Solution by Kenneth A. Ribet, Brown University.

Let $\epsilon>0$ be given. Consider an $\epsilon'<\epsilon$ such that ϵ' is positive and both $f(x_0+\epsilon')$ and $f(x_0-\epsilon')$ are defined. For convenience, define $f(x)=0$ for $x\notin[a, b]$. Then define:

$$f(x_0 + \epsilon) - f(x_0) = \delta_1 > 0,$$
$$f(x_0) - f(x_0 - \epsilon) = \delta_2 > 0,$$
$$\delta = \min(\delta_1, \delta_2).$$

Clearly $|f(x)-f(x_0)|<\delta$ implies $|x-x_0|<\epsilon'$, so that f^{-1} is continuous at x_0.

Also solved by the proposer.

Rhymed Square

631. [September, 1966] *Proposed by J. A. H. Hunter, Toronto, Ontario, Canada.*

My first and last
add up to four.
My middle two
to just eight more.

My second from
my fifth makes three.
I'm square! What must
my whole six be?

Solution by Anton Glaser, Pennsylvania State University, Ogontz Campus.

Let $abcdef$ be the six digits of the number N. Since N is square, $f \in\{0, 1, 4, 5, 6, 9\}$. Now $a\neq 0$ and $a+f=4$ yields $a=3$ and $f=1$. Either one of the two conditions $c+d=12$ and $e-b=3$ would now suffice to pick the *unique* solution $N=337561$ among the 18 entries in a table of squares that satisfy the other conditions.

The problem brought out the poetic impulses of the following solvers:

R. H. Anglin, Danville, Virginia.

Make a square
of five-eight-one.
You'll have my six
when you are done.

Michael Goldberg, Washington, D. C.

If six digits I'm to be,
The digits in my root are three.
Since four is sum of first and last,
And nought as first cannot be passed,
And squares can't end in three or two,
My last as one or nought might do.
A list of squares will quickly show
That final nought we must forego,
And my six digits, when all done,
Are three three seven, five six one.

Sidney Kravitz, Dover, New Jersey.

If first and last add up to four,
Then last is nought or one, none more.
Since two or three for last can't be,
This makes the first a four or three.
A chart of n square now will show,
That you do not have far to go.
You take the square of five eight one,
It's three three seven five six one.

Herbert R. Leifer, Pittsburgh, Pennsylvania.

Add three to one
to get the four.
Add seven to five
to get eight more.
Take three from six
and find the three.
Square five eight one
and all six see.

Julius P. Ordoña, Iowa State University.

Three, three, seven,
five, six and one,
The number search
wasn't so much fun.
For tables square
are always there,
But rhymes are rare
for all math's care.

Stanley Rabinowitz, Far Rockaway, New York.

Since your fifth is greater than three,
Neither nought nor four may your last be.
Hence you end in one,
And the previous one
Must either eight, six, or four be.
So you start with three-five, three-three,
or three-one;
And now that all of this is done,
A table of squares do we consult,
To come up with the final result
Of three-three-seven-five-six-one.

Also solved by Merrill Barnebey, Wisconsin State University at LaCrosse; C. R. Berndtson, M.I.T. Lincoln Laboratory; Sarah Brooks, Utica Free Academy, New York; R. J. Cormier, Northern Illinois University; Jack Dix, Rutgers University; David Fettner, City College of New York; Richard A. Jacobson, Houghton College, New York; Michael J. Martino, Pennsylvania State University, Ogontz Campus; John W. Milsom, Slippery Rock State College; William L. Mrozek, University of Michigan; Michael W. O'Donnell, University of Missouri; C. C. Oursler, Southern Illinois University (Edwardsville); Richard Riggs, Jersey City State College; Marilyn R. Rodeen, San Francisco, California; Lawrence E. Schaefer, General Motors Institute, Flint, Michigan; Jerome J. Schneider, Chicago, Illinois; Sister M. Stephanie Sloyan, Georgian Court College, New Jersey; Paul Sugarman, Swampscott High School, Massachusetts; John H. Tiner, Harrisburg, Arkansas; Charles W.Trigg, San Diego, California; Lowell T. Van Tassel, San Diego City College; Charles R. Wall, Knoxville, Kentucky; Hazel S. Wilson, St. Petersburg, Florida; Charles Ziegenfus, Madison College, Virginia; and the proposer.

A Well-Known Sum

632. [September, 1966] *Proposed by Erwin Just, Bronx Community College.*

Prove that

$$1+2\sum_{n=1}^{\infty}\frac{(-1)^n}{1-(n\pi)^2}=\frac{1}{\sin 1}\,.$$

Solution by Eldon Hansen, Lockheed Research Laboratory, Palo Alto, California.

The solution to this problem is well known. The equation

$$(1)\qquad \frac{1}{x^2}+2\sum_{n=0}^{\infty}\frac{(-1)^n}{x^2-(n\pi)^2}=\frac{1}{x}\csc x$$

is either stated or proved as equation 4.3.93 of [**1**], equation 6.495(4) of [**2**], example (15) page 225 of [**3**], page 296 of [**3**], equation (5) page 361 of [**4**], equations (796) and (772) of [**5**], equation (9) page 20 of [**6**], equations 1.422(3) and 1.422(5) of [**7**], and equation (84) page 210 of [**8**].

Letting $x=1$ in (1) yields the desired result.

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1. Abramowitz and Stegun, Handbook of Mathematical Functions, N.B.S. Applied Math. Series, #55, 1964.

2. Adams and Hippisley, Smithsonian Mathematical Formulae and Tables of Elliptic Functions, Smithsonian Institute, 1947.

3. T. J. Bromwich, An Introduction to the Theory of Infinite Series, Macmillan, 1947.

4. G. Chrystal, Algebra, An Elementary Textbook, Part II, Dover, 1961.

5. L. B. W. Jolley, Summation of Series, 2nd rev. ed., Dover, 1961.

6. V. Mangulis, Handbook of Series for Scientists and Engineers, Academic Press, 1965.

7. Ryshik and Gradstein, Tables of Series, Products, and Integrals, Deutscher Verlag der Wissenschaften, 1963.

8. I. J. Schwatt, An Introduction to the Operations with Series, 2nd ed., Chelsea, 1924.

Also solved by Wray G. Brady, University of Bridgeport, Connecticut; Dennis E. Delorey, Newtonville, Massachusetts; Harvey J. Fletcher, Brigham Young University; Michael Goldberg, Washington, D. C.; Simon Green, California State Polytechnic College, Pomona, California; Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; Sidney H. Kung, Jacksonville University, Florida; Charles R. Wall, Knoxville, Tennessee; and the proposer.

The proposer showed how this problem followed from the solution to Problem 559, this MAGAZINE, September, 1964.

An Irrational Sum

633. [September, 1966] *Proposed by Dov Avishalom, Hebrew University, Jerusalem.*

Prove that the number

$$(1965^{1966} + 1968^{1967})^{0.5}$$

is irrational.

Solution by Mannis Charosh, New Utrecht High School, Brooklyn, New York.

If it can be shown that the number in parenthesis is not an integral square, the theorem will be proven.

We have

$$(1965)^{1966} \equiv 5 (\text{mod } 10)$$

and

$$(1968)^{1967} \equiv 8^{4.491+3} \equiv 2 (\text{mod } 10).$$

Therefore

$$(1965)^{1966} + (1968)^{1967} \equiv 7 (\text{mod } 10).$$

Since 7 is not a quadratic residue modulo 10, the required result follows.

Also solved by P. N. Bajaj, Western Reserve University; Merrill Barnebey, Wisconsin State University at La Crosse; C. R. Berndtson, M.I.T. Lincoln Laboratory; Sarah Brooks, Utica Free Academy, New York; Raphael T. Coffman, Richland, Washington; R. J. Cormier, Northern Illinois University; Jack Dix, Rutgers University; Anton Glaser, Pennsylvania State University, Ogontz Campus; Michael Goldberg, Washington, D. C.; Simon Green, California State Polytechnic College, Pomona, California; J. A. H. Hunter, Toronto, Ontario, Canada; Richard A. Jacobson, Houghton College, New York; M. Kariman, Brooklyn, New York; Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; Vivian Koffer and Roy Kligfield (jointly), Flushing High School, New York; Sidney Kravitz, Dover, *New Jersey; Herbert R. Leifer, Pittsburgh, Pennsylvania; Douglas Lind, University of Virginia; Michael J. Martino, Pennsylvania State University, Ogontz Campus; Lawrence V. Novak, Pennsylvania State University; Stanley Rabinowitz, Far Rockaway, New York; Kenneth A. Ribet, Brown University; Jerome J. Schneider, Chicago, Illinois; David L. Silverman, Hughes Aircraft Co., El Segundo, California; Paul Sugarman, Swampscott High School, Massachusetts; Lowell T. Van Tassel, San Diego City College; Charles Ziegenfus, Madison College, Virginia; and the proposer.*

A Polynomial of Primes

634. [September, 1966] *Proposed by R. S. Luthar and Stephen Wurzel, Colby College, Maine.*

If p is a prime, such that

$$p^2 \not\equiv p \pmod 3$$

show that

$$p^{2n-1} + p^{2n-3} + \cdots + p + n \equiv 0 \pmod 3.$$

Solution by Stanley Rabinowitz, Far Rockaway, New York.

The conclusion is true if p is any number relatively prime to 3. If $p^2-p\not\equiv 0 \pmod 3$ and $(p, 3)=1$, then $p\not\equiv 0 \pmod 3$, and so $p-1\not\equiv 0 \pmod 3$. The only other case is that $p\equiv -1 \pmod 3$. Since by Fermat's Theorem, $p^2\equiv 1 \pmod 3$, we also have $p^3\equiv -1$, $p^5\equiv -1$, $p^7\equiv -1, \cdots, \pmod 3$. Hence $p^{2n-1}+p^{2n-3}+\cdots p^3+p\equiv(-1)+(-1)+\cdots+(-1)+(-1)\equiv -n \pmod 3$ and the result follows.

Also solved by R. H. Anglin, Danville, Virginia; P. N. Bajaj, Western Reserve University; Merrill Barnebey, Wisconsin State University at La Crosse; Sister Marion Beiter, Rosary Hill College, New York; Arthur Bolder, Brooklyn, New York; Nicholas C. Bystrom, St. Paul, Minnesota; Mannis Charosh, Brooklyn, New York; Chinthayamma, University of Alberta, Canada; R. J. Cormier, Northern Illinois University; Mickey Dargitz, Ferris State College, Michigan; Jack Dix, Rutgers University; J. D. Featherstone, University of Southern California; William F. Feeny, University of Pittsburgh; Tadashi Fujiwara, Hyogo-Ken, Japan; Simon Green, California State Polytechnic College, Pomona, California; Michael Goldberg, Washington, D. C.; Harry R. Henshaw, Victoria, B.C.; Aughtum S. Howard, Eastern Kentucky University; Richard A. Jacobson, Houghton College, New York; Erwin Just and Norman Schaumberger (jointly), Bronx Community College; Victor H. Keiser, Jr., Whitman College, Washington; Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; Douglas Lind, University of Virginia; Edward Moylan, University of Wisconsin; Kenneth A. Ribet, Brown University; Richard Riggs, Jersey City State College; Marilyn R. Rodeen, San Francisco, California; Jerome J. Schneider, Chicago, Illinois; Jay Spitzen, Abraham Lincoln High School, Brooklyn, New York; Charles W. Trigg, San Diego, California; Charles R. Wall, Knoxville, Tennessee; Donald R. Wilder, Rochester, New York; Charles Ziegenfus, Madison College, Virginia; and the proposers.

Comment on Problem 602

602. [November, 1965, and May, 1966] *Proposed by Bruce W. King, SUNY at Buffalo, New York.*

Show that

$$\sum_{i=0}^{n-2} \binom{n}{i} (-x)^{n-i}(x+\lambda)^i(n-i-1)$$

gives all but the last two terms of the expansion of $(x+\lambda)^n$.

Comment by Henry W. Gould, West Virginia University.

The two published solutions of Problem 602 (this MAGAZINE, May, 1966) tend to obscure the simplicity of the result in that the first invokes differentiation techniques and the second uses matrix theory and results quoted from elsewhere. It may therefore be of interest to exhibit a solution using only series operations, the binomial theorem, and the easy identity

$$(n-k)\binom{n}{k} = n\binom{n-1}{k}.$$

We have

$$\begin{aligned}
\sum_{k=0}^{n-2}\binom{n}{k}&(-x)^{n-k}(x+y)^k(n-k-1)\\
&= \sum_{k=1}^{n-2}\binom{n}{k}(n-k)(-x)^{n-k}(x+y)^k\\
&\quad - \sum_{k=0}^{n-2}\binom{n}{k}(-x)^{n-k}(x+y)^k\\
&= n\sum_{k=0}^{n-2}\binom{n-1}{k}(-x)^{n-k}(x+y)^k\\
&\quad - \sum_{k=0}^{n}\binom{n}{k}(-x)^{n-k}(x+y)^k + n(-x)(x+y)^{n-1} + (x+y)^n\\
&= -nx\sum_{k=0}^{n-1}\binom{n-1}{k}(-x)^{n-k-1}(x+y)^k\\
&\quad - n(-x)(x+y)^{n-1} - y^n + n(-x)(x+y)^{n-1} + (x+y)^n\\
&= -nxy^{n-1} - y^n + (x+y)^n
\end{aligned}$$

as desired to show. All of the steps have been given, but several may be easily omitted as obvious.

Comment on Problem 611

611. [January and September, 1966] *Proposed by A. Struyk, Paterson, New Jersey.*

A well known problem is that in which a rectangular sheet of given dimensions (length L, width W) is to have cut from its corners a square (side x) so that, when the resulting figure is folded to form an open box, the box will have maximum volume.

Comment by Charles W. Trigg, San Diego, California.

The solution published on Page 251 of the September, 1966, issue, starts with an incorrect expression for the volume. It is given that the squares cut from the corners have sides x, so the volume is

$$V = x(L - 2x)(W - 2x).$$

Also, the (a) part of the question is not considered, nor is a one-parameter solution of (b) exhibited.

Now,

$$dV/dx = 12x^2 - 4x(L + W) + LW = 0.$$

So,

$$x = (L + W - \sqrt{L^2 - LW + W^2})/6$$

for maximum volume. Then

$$L = p^2 - q^2,$$
$$W = 2pq - q^2$$

leads to

$$x = q(p - q)/2.$$

(a) Minor manipulation of the arithmetic progression,

$$A = q + p, \qquad B = q, \qquad C = q - p, \qquad D = q - 2p$$

leads to

$$L = -AC, \qquad W = -BD, \qquad x = -BC/2.$$

(b) Let $p=2q+1$, then

$$x = q(q + 1)/2, \qquad L - 2x = (2q + 1)(2q + 2),$$
$$W - 2x = 2q(2q + 1)/2$$

and all of these dimensions obviously are triangular numbers.

Thus if the first three triangular numbers be set down in the first column of a table, the series of consecutive triangular numbers in the first row, and the consecutive triangular numbers alternately in the second and third rows, then the columnar triads constitute the dimensions of maximum volume boxes.

q	1	2	3	4	5	6	7 · · ·
x	1	3	6	10	15	21	28 · · ·
W	3	10	21	36	55	78	105 · · ·
1	6	15	28	45	66	91	120 · · ·
W	5	16	33	56	85	120	161 · · ·
L	8	21	40	65	96	133	176 · · ·

By adding $2x$ to w and to 1, the dimensions of the original sheet are obtained. This method gives an entirely different set of values from those in the other solution.

Erratum: In the solution to Problem 625, January 1067, page 49, the exponent of q in the "correct" formula product should be 2^k instead of $2k$.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q404. The roots of $2x^3-6x^2+8x-3=0$ are r_1, r_2, r_3. Find the value of the expression

$$r_1 + r_2 + r_3 + \frac{1}{r_1r_2} + \frac{1}{r_2r_3} + \frac{1}{r_3r_1}\cdot$$

[*Submitted by Charles W. Trigg*]

Q405. It is apparent that a bounded figure need not have a unique chord of maximum length. Show, however, that two such maximum chords cannot be parallel.

[*Submitted by Murray S. Klamkin*]

Q406. Let

$$S_n = \sum_{k=0}^{n} 1/k\cdot$$

Given an integer n, find an integer m, $m>n$, so that $S_m-S_n>1$.

[*Submitted by John Beidler*]

Q407. Prove that

$$e = 2 \exp \sum_{n=1}^{\infty} (\ln 2)^{n-1}/n!$$

[*Submitted by Dan Sonnenschein*]

Q408. A tetrahedron has each pair of nonintersecting sides equal: each of one pair is $\sqrt{5}$ in length, each of another pair is $\sqrt{10}$ in length, and each of the third is $\sqrt{13}$. What is the volume of the tetrahedron?

[*Submitted by Alan Sutcliffe*]

(Answers on page 85)

is a semisimple ring (zero Jacobson radical) if and only if in Z_{np} there are no nilpotent elements of the form yp for y in Z.

References

1. L. Fuchs, Abelian Groups, Pergamon Press, New York, 1960.
2. Marshall Hall, Jr., Theory of Groups, Macmillan, New York, 1959.
3. R. E. Peinado, Elementos nilpotentes e idempotentes en los Anillos Z_n, Rev. Mat. Hisp-Amer., Madrid, Vol. XXVI (1966), pp. 42–46.

ANSWERS

A404. The $1/r_i$ are the roots of the reciprocal equation $3y^3-8y^2+6y-2=0$. Hence, by the relationships between the roots and coefficients, the given expression equals $6/2+6/3$ or 5. Also, the given expression is equivalent to $(r_1+r_2+r_3)(1+1/r_1r_2r_3)$, so is equal to $(6/2)\ [1+1/(3/2)]$ or 5.

A405. The proof is indirect. Assume two congruent and parallel chords of maximum length. The endpoints of these chords are the vertices of a parallelogram, one of whose diagonals, at least, is larger than all the sides. This contradicts our initial assumption and, consequently, we obtain our stated result.

A406. Using the comparison test for series, one finds that

$$\ln (n + 1) = \int_1^{n+1} (1/x)dx < S_n.$$

So now we have

$$S_m - S_n > \ln(m + 1) - \ln(n + 1) = \ln \frac{m + 1}{n + 1}$$

and $\ln m+1/n+1>1$ whenever $m+1/n+1>e$. So by picking $m>(n-1)e-1$ we obtain the desired result. To simplify the choice of m, since $3>e$, the choice $m=3n+2$ will do.

A407.

$$\begin{aligned} e^{\ln 2} &= 2 \exp \sum_{n=1}^{\infty} \frac{(\ln 2)^n}{n!} \\ &= 2 \exp \sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!} - 1 \\ &= 2 \exp e^{\ln 2} - 1 \\ &= 2. \end{aligned}$$

A408. Taking the rectangular solid with sides 1, 2, and 3, and volume 6, the lengths of the diagonals of its face are $\sqrt{5}$, $\sqrt{10}$, and $\sqrt{13}$. The tetrahedron can be formed by removing four corners of this rectangular body, cutting along diagonals. The volume of the tetrahedron is 2.

(Quickies on page 110)

Important Texts

TRIGONOMETRY: AN ANALYTIC APPROACH

By Irving Drooyan and Walter Hadel, both of Los Angeles Pierce College

Based on recent educational trends in mathematics, this textbook provides a completely modern introduction to trigonometry; emphasis is on topics particularly important to students who need a good background in mathematics. A Teacher's Manual, Solutions Manual, Progress Tests, and Answers to Progress Tests are available.

1967, approx. 416 pages, $5.50

MODERN PLANE GEOMETRY FOR COLLEGE STUDENTS

By Herman R. Hyatt and Charles C. Carico, both of Los Angeles Pierce College

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By Helen L. Garstens and Stanley B. Jackson, both of the University of Maryland

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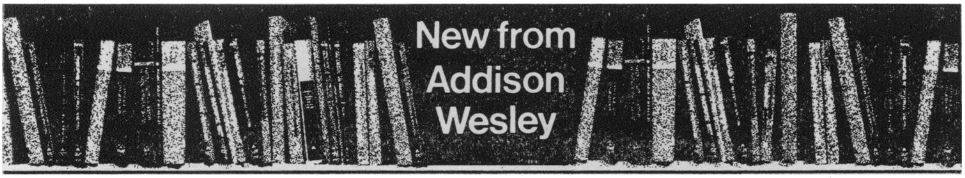
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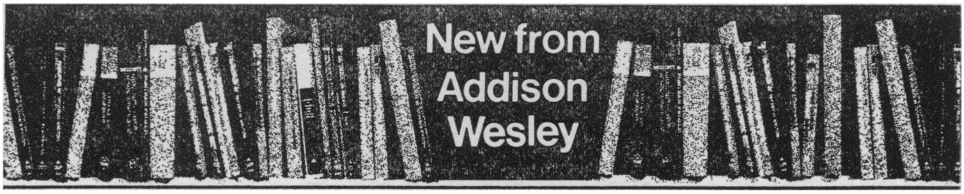

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